13.3, Part Three) Normal and Binormal Vectors, and Osculating Circles and Planes

1. The Principal Unit Normal Vector:

$\mathbf{T}'(t)$ is a vector-valued function. It can be equal to the zero vector at a particular value of t, in which case $|\mathbf{T}'(t)| = 0$ at that instant.

It is even possible that we could have $\mathbf{T}'(t) = \mathbf{0}$ and $|\mathbf{T}'(t)| = 0$ for *all* values of t. This occurs in the case of *linear motion*. When our particle is moving linearly, $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{v}$ is a fixed nonzero vector. If this is the case, then for all t, $\mathbf{v}(t) = \mathbf{v}$, $v(t) = v = |\mathbf{v}|$, $\mathbf{a}(t) = \mathbf{0}$, $a(t) = 0$, $\mathbf{T}(t) = \frac{\mathbf{v}}{v}$, $\mathbf{T}'(t) = \mathbf{0}$, and $|\mathbf{T}'(t)| = 0$.

For a particle moving nonlinearly, $\mathbf{T}'(t)$ and $|\mathbf{T}'(t)|$ can be zero at certain values (but not all values) of t. This occurs if, for certain values of t, the particle has nonzero speed but zero acceleration Since $\mathbf{T}'(t) = v(t)^{-3}\left[v(t)^2\mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)\right]$, any such value of t would give us $v(t)^{-3}\left[v(t)^2\mathbf{0} - \mathbf{v}(t)\cdot\mathbf{0}\ \mathbf{v}(t)\right] = v(t)^{-3}\left[\mathbf{0} - 0\ \mathbf{v}(t)\right] = v(t)^{-3}[\mathbf{0} - \mathbf{0}] = v(t)^{-3}\mathbf{0} = \mathbf{0}$.

Here is an example in two dimensions: Suppose $\mathbf{r}(t) = < t^3, \sin t >$. Then $\mathbf{v}(t) = < 3t^2, \cos t >$, $v(t) = \sqrt{9t^4 + \cos^2 t}$, $\mathbf{a}(t) = < 6t, -\sin t >$, $a(t) = \sqrt{36t^2 + \sin^2 t}$. Notice that $v(t)$ is never zero, so $\mathbf{T}(t)$ is defined for all t, namely, $\mathbf{T}(t) = (9t^4 + \cos^2 t)^{-1/2} < 3t^2, \cos t >$. At $t = 0$, we have acceleration $\mathbf{0}$, so $\mathbf{T}'(t) = \mathbf{0}$, based on the above analysis. However, just to confirm this, let's go ahead and find $\mathbf{T}'(t)$.

Since $\mathbf{T}'(t) = v(t)^{-3}\left[v(t)^2\mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)\right]$, we get
$(9t^4 + \cos^2 t)^{-3/2}[(9t^4 + \cos^2 t) < 6t, -\sin t > -(18t^3 - \cos t \sin t) < 3t^2, \cos t >]$,
which simplifies to
$(9t^4 + \cos^2 t)^{-3/2} < 6t\cos^2 t + 3t^2 \cos t \sin t, -9t^4 \sin t - 18t^3 \cos t >$, or
$(9t^4 + \cos^2 t)^{-3/2} < 3t \cos t\ (2\cos t + t \sin t), -9t^3(t \sin t + 2\cos t) >$, or
$3t(9t^4 + \cos^2 t)^{-3/2} < \cos t\ (2\cos t + t\sin t), -3t^2(t\sin t + 2\cos t) >$.
When $t = 0$, we get $< 0, 0 >$.

When $\mathbf{T}'(t)$ is nonzero, it has a positive magnitude and it has a direction. As already discussed, the former is the particle's *speed of direction change*. This is a real-valued measure of *how quickly* the particle is changing direction. On the other hand, the direction of $\mathbf{T}'(t)$ indicates *the direction in which the particle is changing direction*. In other words, it tells us *the direction in which the direction is changing* at a given instant.

We deal with the *unit tangent vector* instead of *velocity* in situations where we are not concerned with the *speed of motion*, but are interested only in the *direction of motion*. Analogously, in some situations, we are not concerned with the *speed of direction change*, but are interested only in the *direction of direction change*.

To analyze the *direction* of motion without heed to the *speed* of motion, we study the *unit tangent vector*, which is the unit vector having the same direction as velocity. To analyze the *direction* of direction change without heed to the *speed* of direction change, we do something similar: We study the unit vector having the same direction as $\mathbf{T}'(t)$. To obtain this, we divide $\mathbf{T}'(t)$ by its own magnitude (just as we obtained $\mathbf{T}(t)$ by dividing $\mathbf{v}(t)$ by its own magnitude). We call this the **principal unit normal vector**, and we denote it by $\mathbf{N}(t)$. Thus, $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. This vector is *undefined* when $\mathbf{T}'(t) = \mathbf{0}$, just as $\mathbf{T}(t)$ is undefined when $\mathbf{v}(t) = \mathbf{0}$.

The name "principal unit normal vector" has been very carefully chosen. Let's review each of the terms included in this name.

- $\mathbf{N}(t)$ is a *unit vector*, i.e., a vector of length one.
- Since $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$, are since $\mathbf{N}(t)$ has the same direction as $\mathbf{T}'(t)$, it follows that $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$. In the context of nonzero vectors, "orthogonal" means "perpendicular," and another word for "perpendicular" is *normal*.
- For two-dimensional motion, at any value of t, there are *two* different unit vectors that are normal to $\mathbf{T}(t)$. One of them is $\mathbf{N}(t)$ and the other is $-\mathbf{N}(t)$. Both of these are *unit normal vectors*, but the former is designated the *principal* unit normal vector. This is analogous to basic arithmetic. Any positive real number k has two square roots, which are opposites of each other. The positive root is denoted $\sqrt{k}$ and the negative root is denoted $-\sqrt{k}$. Both of these are *square roots* of k, but the former is designated the *principal* square root of k.
- For three-dimensional motion, the situation is more complicated. At any value of t, there are *infinitely many* different unit vectors that are normal to $\mathbf{T}(t)$. Among this infinite collection are $\mathbf{N}(t)$ and its opposite, $-\mathbf{N}(t)$. All of these are *unit normal vectors*, but $\mathbf{N}(t)$ is designated the *principal* unit normal vector.

Since $\mathbf{T}' = v^{-3}\left[\mathbf{v} \cdot \mathbf{v}\ \mathbf{a} - \mathbf{v} \cdot \mathbf{a}\ \mathbf{v}\right]$ and $|\mathbf{T}'| = v^{-3}\left|\mathbf{v} \cdot \mathbf{v}\ \mathbf{a} - \mathbf{v} \cdot \mathbf{a}\ \mathbf{v}\right|$, it follows that $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{\mathbf{v}\cdot\mathbf{v}\ \mathbf{a} - \mathbf{v}\cdot\mathbf{a}\ \mathbf{v}}{\left|\mathbf{v}\cdot\mathbf{v}\ \mathbf{a} - \mathbf{v}\cdot\mathbf{a}\ \mathbf{v}\right|}$. In other words, v^{-3} will cancel out.

Alternatively, since $\mathbf{T}' = v^{-2}[v\mathbf{a} - v'\mathbf{v}]$ and $|\mathbf{T}'| = v^{-2}|\mathbf{v} \times \mathbf{a}|$, it follows that $\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|} = \frac{v\mathbf{a} - v'\mathbf{v}}{|\mathbf{v} \times \mathbf{a}|}$. In other words, v^{-2} will cancel out.

In Section 13.2, we studied the helix $\mathbf{r}(t) = < \cos t, \sin t, t^3 >$. Finding $\mathbf{N}(t)$ for this curve is exceedingly difficult, so let us instead consider a simpler helix, $\mathbf{r}(t) = < \cos t, \sin t, t >$. Now we have $\mathbf{v}(t) = < -\sin t, \cos t, 1 >$, $\mathbf{a}(t) = < -\cos t, -\sin t, 0 >$, $v(t) = \sqrt{2}$, $\mathbf{T}(t) = \frac{1}{\sqrt{2}} < -\sin t, \cos t, 1 >$, $\mathbf{T}'(t) = \frac{1}{\sqrt{2}} < -\cos t, -\sin t, 0 >$, $|\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$, and $\mathbf{N}(t) = < -\cos t, -\sin t, 0 >$. Notice that in this case, $\mathbf{N}(t) = \mathbf{a}(t)$.

In general, when speed of motion is a constant, v, then $\mathbf{T}'(t) = \frac{1}{v}\mathbf{a}(t)$. If, furthermore, acceleration is a unit vector, then $|\mathbf{T}'(t)| = \frac{1}{v}$, in which case $\mathbf{N}(t) = \mathbf{a}(t)$.

2. The Normal Line, the Osculating Circle, and the Osculating Plane:

$\mathbf{N}(t)$ serves as the direction vector for a special line passing through the point of tangency, P_t, that is perpendicular to the tangent line. This line is called the **normal line**.

$\mathbf{N}(t)$ points in the direction in which the curve is turning, also referred to as the **direction of curvature**.

At the point of tangency, instead of approximating the curve with a *tangent line*, we can instead approximate the curve with a *tangent circle*, which is formally known as the **osculating circle**. This circle lies *on the concave side* of the curve, and $\mathbf{N}(t)$ points toward the circle's center (the center lies on the normal line). At the point of tangency, the curve of motion and the osculating circle have the *same tangent line* and the *same curvature*. Consequently, the osculating circle is also known as the **circle of curvature**, and its center is known as the **center of curvature**.

As we know from Calculus I, if a function is linear, then the graph of the function is its own tangent line. Similarly, if a curve is a circle, then the curve is its own osculating circle.

The preceding paragraphs apply equally well to either two-dimensional motion or three-dimensional motion. The following discussion deals specifically with the latter...

At the point of tangency, the curve fits into (or comes infinitesimally close to fitting into) a unique plane which contains the tangent line, the normal line, and the tangent circle. This plane is referred to as the **osculating plane**.

(By the way, the word "osculating" literally translates as "kissing." The osculating plane and the osculating circle are so named because they are said to *kiss* the curve at the point of tangency. In fact, the tangent line could be referred to as the *osculating line*, because it likewise "kisses" the curve at the point of tangency.)

3. The Binormal Vector and the Normal Plane:

To write the equation of the osculating plane, we need a vector perpendicular to the plane. The vector we use is $\mathbf{T}(t) \times \mathbf{N}(t)$, which is known as the **binormal vector** and is denoted $\mathbf{B}(t)$. This is also a unit vector (i.e., length 1). (In general, the cross product of perpendicular unit vectors must be a unit vector.)

$\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ are unit vectors and a pairwise-perpendicular. In this regard, they are exactly like the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. We usually picture the latter three vectors as being placed in standard position, i.e., so their tail is the origin, whereas we picture the former three vectors as being placed so their tail is P_t, the point of tangency. When so placed, the osculating plane contains $\mathbf{T}(t)$ and $\mathbf{N}(t)$, while $\mathbf{B}(t)$ serves as the normal vector for the plane. Also of interest is the plane containing $\mathbf{N}(t)$ and $\mathbf{B}(t)$, for which $\mathbf{T}(t)$ serves as the normal vector. This plane is known as the **normal plane**. The curve of motion perpendicularly “pierces through” the normal plane at the point P_t.

The osculating plane and the normal plane are perpendicular to each other, and their intersection is the normal line.

In our previous example, we had the helix $\mathbf{r}(t) = < \cos t, \sin t, t^3 >$, where $\mathbf{T}(t) = \frac{1}{\sqrt{2}} < -\sin t, \cos t, 1 >$ and $\mathbf{N}(t) = < -\cos t, -\sin t, 0 >$. In this case, we get $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}}\{< -\sin t, \cos t, 1 > \times < -\cos t, -\sin t, 0 >\} = \frac{1}{\sqrt{2}} < \sin t, -\cos t, 1 >$.

4. Miscellaneous Theorems:

Recall that $\mathbf{N}$ is the unit vector in the direction of $\mathbf{T}'$, i.e., $\frac{\mathbf{T}'}{|\mathbf{T}'|}$, and hence $\mathbf{N}$ and $\mathbf{T}'$ are scalar multiples of each other. Specifically, $\mathbf{N} = \frac{1}{|\mathbf{T}'|}\mathbf{T}'$ and $\mathbf{T}' = |\mathbf{T}'|\mathbf{N}$. But since $|\mathbf{T}'| = v\kappa$, we can write $\mathbf{N} = \frac{1}{v\kappa}\mathbf{T}'$ and $\mathbf{T}' = v\kappa\mathbf{N}$.

For $v \neq 0$, $\mathbf{a} = v'\mathbf{T} + v^2\kappa\mathbf{N}$.
Proof: Since $\mathbf{a} = v'\mathbf{T} + v\mathbf{T}'$ and $\mathbf{T}' = v\kappa\mathbf{N}$, $\mathbf{a} = v'\mathbf{T} + v(v\kappa\mathbf{N}) = v'\mathbf{T} + v^2\kappa\mathbf{N}$.

Since acceleration is a linear combination of $\mathbf{T}$ and $\mathbf{N}$, both of which lie in the osculating plane, the acceleration vector *itself* lies in the osculating plane.

v' is known as the **tangential component of acceleration**, and is denoted a_T.

$v^2\kappa$ is known as the **normal component of acceleration**, and is denoted a_N.

Thus, we have $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$, where $a_T = v'$ and $a_N = v^2\kappa$.

For $v \neq 0$, $\mathbf{v} \times \mathbf{a} = v^3\kappa\mathbf{B}$
Proof: Since $\mathbf{v} \times \mathbf{a} = v^2(\mathbf{T} \times \mathbf{T}')$ and $\mathbf{T}' = v\kappa\mathbf{N}$,
$\mathbf{v} \times \mathbf{a} = v^2(\mathbf{T} \times v\kappa\mathbf{N}) = v^2 v\kappa(\mathbf{T} \times \mathbf{N}) = v^3\kappa(\mathbf{T} \times \mathbf{N}) = v^3\kappa\mathbf{B}$.

We have already shown that $v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$, so $a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$

$a_N = \frac{|\mathbf{v} \times \mathbf{a}|}{v}$
Proof: Since $a_N = v^2\kappa$ and $\kappa = v^{-3}|\mathbf{v} \times \mathbf{a}|$,
$a_N = v^2 v^{-3}|\mathbf{v} \times \mathbf{a}| = v^{-1}|\mathbf{v} \times \mathbf{a}| = \frac{|\mathbf{v} \times \mathbf{a}|}{v}$