5.3 The Fundamental Theorem of Calculus (FTC)

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. The Fundamental Theorem of Calculus give the precise inverse relationship between the derivative and the integral.

The Fundamental Theorem of Calculus, Part 1: IF *f* is continuous on [a, b], **THEN,** the function ***g*** defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b) and $g'(x) = f(x)$.

Using Leibniz notation for derivatives, we can write FTC1 as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example: Find the derivative of the function using FTC1

a) $F(x)=\int_x^0 \sqrt{1+\sec(t)}dt$ $\quad \left[Hint:\ \int_x^0 \sqrt{1+\sec(t)}dt = -\int_0^x \sqrt{1+\sec(t)}dt\right]$

$$F(x) = -\int_0^x \sqrt{1+\sec(t)}\,dt$$

$$\boldsymbol{F'(x) = -\sqrt{1+\sec(x)}}$$

(***Roughly*** speaking, FTCI says that if we first integrate ***f*** then differentiate the result, we get back to the original function ***f***. However, when the upper limit "x" is a more involved function, we must use the chain-rule and use the derivative of the upper limit.)

b) $y = \int_1^{3x+2} \frac{t}{1+t^3}dt$

Let u = 3x + 2, then

$$y' = \frac{d}{dx}\int_1^{3x+2}\frac{t}{1+t^3}dt = \frac{d}{dx}\int_1^{u}\frac{t}{1+t^3}dt = \frac{d}{dx}\left[\int_1^{u}\frac{t}{1+t^3}dt\right]\frac{du}{dx} \leftarrow chain\ rule$$

$$y' = \frac{u}{1+u^3}\cdot\frac{du}{dx} = \frac{(3x+2)}{1+(3x+2)}\cdot 3 = \boldsymbol{\frac{3(3x+2)}{1+(3x+2)}}$$

The Fundamental Theorem of Calculus, Part 2: IF *f* is continuous on [a, b], **THEN,**

$$\boldsymbol{\int_a^b f(x)dx = F(b) - F(a)}$$

where **F** is any antiderivative of ***f***, that is, a function such that **F' = *f***.

Example: Evaluate the integrals:

a) $\int_1^3 (x^2+2x-4)dx$

b) $\int_{\frac{\pi}{6}}^{\pi} \sin(\theta)d\theta$

a) $\int_1^3 (x^2+2x-4)dx = \mathrm{F}(3) - \mathrm{F}(1)$, where $F(x) = \frac{x^3}{3} + x^2 - 4x$ so the equation can be written as

$$\int_1^3 (x^2+2x-4)dx = F(x)]_1^3 = \left(\frac{3^3}{3} + 3^2 - 4(3)\right) - \left(\frac{1^3}{3} - 1^2 - 4(1)\right) =$$

$$\left(\frac{27}{3} + 9 - 12\right) - \left(\frac{1}{3} - 1 - 4\right) =$$

$$(9+9-12) - \left(\frac{1}{3} - 1 - 4\right) =$$

$$6 - \left(-\frac{14}{3}\right) = 6 + \frac{14}{3} = \mathbf{\frac{32}{3}}$$

b) $\int_{\frac{\pi}{6}}^{\pi} \sin(\theta)d\theta$: Since the antiderivative of $\boldsymbol{\sin\theta} = -\boldsymbol{\cos\theta}$, we have

$$\int_{\frac{\pi}{6}}^{\pi} \sin(\theta)d\theta = -cos]_{\pi/6}^{\pi} = -\cos(\pi) - \left(-\cos\left(\frac{\pi}{6}\right)\right) = -\cos(\pi) + cos\left(\frac{\pi}{6}\right) = -(-1) + \frac{\sqrt{3}}{2} = \mathbf{\frac{2+\sqrt{3}}{2}}$$

We now bring together Part 1 and Part 2 of the theorem:

The Fundamental Theorem of Calculus: Suppose *f* is continuous on [a, b].

1. If g(x) = $\int_a^x f(t)dt$, then $g'(x) = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of *f*, that is, F ' = *f*.

This is the most important theorem of Calculus.