12.6) Cylinders and Quadric Surfaces

In Section 12.1, we discussed the most basic surfaces: *planes*, *spheres*, and *cylinders* (in particular, *circular cylinders* and *parabolic cylinders*). We noted that any two-dimensional curve can generate a cylinder through the process of orthogonal projection. We also discussed the conditions under which a surface represents a function (specifically, z as a function of x and y). I recommend reviewing Section 12.1, as we will be building upon those concepts here.

In this section, we will discuss two more important types of cylinder: elliptic cylinders and *hyperbolic cylinders*. We will also study a broad category of surfaces known as *quadric surfaces* (the sphere belongs to this category).

1. **Cylinders**:

An **elliptic cylinder** is the cylinder obtained by orthogonally projecting an *ellipse*.

In the x,y plane, an *ellipse* centered at the origin has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (We assume a and b are positive. The x intercepts are a and $-a$, while the y intercepts are b and $-b$. If $a = b$, the ellipse is simply a circle. If $a \neq b$, then the ellipse may be thought of as a "squashed" circle.) Its orthogonal projection into three-dimensional space is a *vertical elliptic cylinder* whose center is the z axis. This surface is not a function.

In the x,z plane, an *ellipse* centered at the origin has the equation $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$. Its orthogonal projection into three-dimensional space is a *horizontal elliptic cylinder* whose center is the y axis. The top half (i.e., the part lying on or above the x,y plane) is a function, $f(x,y) = b\sqrt{1 - \frac{x^2}{a^2}}$. Its domain is $\{(x,y) \mid x \in [-a,a],\ y \in (-\infty,\infty)\}$ and the range is $z \in [0,b]$.

In the y,z plane, an *ellipse* centered at the origin has the equation $\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$. Its orthogonal projection into three-dimensional space is a *horizontal elliptic cylinder* whose center is the x axis. The top half (i.e., the part lying on or above the x,y plane) is a function, $f(x,y) = b\sqrt{1 - \frac{y^2}{a^2}}$. Its domain is $\{(x,y) \mid x \in (-\infty,\infty),\ y \in [-a,a]\}$ and the range is $z \in [0,b]$.

A **hyperbolic cylinder** is the cylinder obtained by orthogonally projecting a *hyperbola*.

In the x,y plane, a *sideways hyperbola* centered at the origin has the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. (We assume a and b are positive. The x intercepts are a and $-a$, and there are no y intercepts. The x axis is the hyperbola's transverse axis. The curve consists of two disjoint branches, which are symmetric to each other over the y axis. The curve has slant asymptotes $y = \pm\frac{b}{a}x$.) Its orthogonal projection into three-dimensional space is a *vertical sideways hyperbolic cylinder* whose center is the z axis. This surface is not a function.

In the x,y plane, an *upright hyperbola* centered at the origin has the equation $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. (We assume a and b are positive. The y intercepts are b and $-b$, and there are no x intercepts. The y axis is the hyperbola's transverse axis. The curve consists of two disjoint branches, which are symmetric to each other over the x axis. The curve has slant asymptotes $y = \pm\frac{b}{a}x$.) Its orthogonal projection into three-dimensional space is a *vertical upright hyperbolic cylinder* whose center is the z axis. This surface is not a function.

In the x,z plane, a *sideways hyperbola* centered at the origin has the equation $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$. Its orthogonal projection into three-dimensional space is a *horizontal sideways hyperbolic cylinder* whose center is the y axis. The top half (i.e., the part lying on or above the x,y plane) is a function, $f(x,y) = b\sqrt{\frac{x^2}{a^2} - 1}$. Its domain is $\{(x,y) \mid x \in (-\infty,-a] \cup [a,\infty),\ y \in (-\infty,\infty)\}$ and the range is $z \in [0,\infty)$.

In the x,z plane, an *upright hyperbola* centered at the origin has the equation $-\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$, or $\frac{z^2}{b^2} - \frac{x^2}{a^2} = 1$. Its orthogonal projection into three-dimensional space is a *horizontal upright hyperbolic cylinder* whose center is the y axis. The top half (i.e., the part lying above the x,y plane) is a function, $f(x,y) = b\sqrt{\frac{x^2}{a^2} + 1}$. Its domain is $\{(x,y) \mid x \in (-\infty,\infty),\ y \in (-\infty,\infty)\}$ and the range is $z \in [b,\infty)$.

In the y,z plane, a *sideways hyperbola* centered at the origin has the equation $\frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$. Its orthogonal projection into three-dimensional space is a *horizontal sideways hyperbolic cylinder* whose center is the x axis. The top half (i.e., the part lying on or above the x,y plane) is a function, $f(x,y) = b\sqrt{\frac{y^2}{a^2} - 1}$. Its domain is $\{(x,y) \mid x \in (-\infty,\infty),\ y \in (-\infty,-a] \cup [a,\infty)\}$ and the range is $z \in [0,\infty)$.

In the y,z plane, an *upright hyperbola* centered at the origin has the equation $-\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$, or $\frac{z^2}{b^2} - \frac{y^2}{a^2} = 1$. Its orthogonal projection into three-dimensional space is a *horizontal upright hyperbolic cylinder* whose center is the x axis. The top half (i.e., the part lying above the x,y plane) is a function, $f(x,y) = b\sqrt{\frac{y^2}{a^2} + 1}$. Its domain is $\{(x,y) \mid x \in (-\infty,\infty),\ y \in (-\infty,\infty)\}$ and the range is $z \in [b,\infty)$.

2. Quadric Surfaces:

Just as a sphere is the three-dimensional version of a circle, an **ellipsoid** is a three-dimensional version of an ellipse. Just as an ellipse can be thought of as a "squashed" circle, an ellipsoid can be thought of as a "squashed" sphere. The equation of an ellipsoid centered at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The "extreme points" of this ellipsoid are the points $(a,0,0)$, $(-a,0,0)$, $(0,b,0)$, $(0,-b,0)$, $(0,0,c)$, and $(0,0,-c)$, which are also its x, y, and z intercepts.

An ellipsoid fails the vertical line test and is not a function. However, if we restrict ourselves to the top half of the ellipsoid–i.e., to the part of the ellipsoid lying on or above the x,y plane–we have a **hemi-ellipsoid**, which *is* a function. The equation of the hemi-ellipsoid is $z = c\sqrt{r^2 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$. If we call this function f, then we may write the formula $f(x,y) = c\sqrt{r^2 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$. The domain of this function is the elliptic disc $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ in the x,y plane. (An **elliptic disc** is an ellipse together with its interior.) (If the point (x,y) were outside this disc, the radicand would be negative and the formula would generate an imaginary result.) The range of this function is the interval $[0,c]$ on the z axis. This kind of function is known as a **hemi-ellipsoidal function**.

Before proceeding to the next type of surface, let us pause to introduce the concept of a **trace**. Generally speaking, the intersection of a surface and a plane will be a curve. (There are exceptions. For example, the intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $z = 1$ is a single point, $(0,0,1)$. The intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $z = 2$ is nothing at all; technically, it's the empty set.) We are particularly interested in three kinds of planes: planes parallel to the x,y plane, which are of the form $z = z_0$, planes parallel to the x,z plane, which are of the form $y = y_0$, and planes parallel to the y,z plane, which are of the form $x = x_0$. The intersection of a surface with any of these types of plane is known as the *trace of the surface* (relative to that plane). In layman's terms, a trace is a *cross-section*.

For example, in the case of the sphere $x^2+y^2+z^2 = 1$, the trace relative to the plane $z = 0.6$ or $z = \frac{3}{5}$ is a circle whose radius is 0.8 or $\frac{4}{5}$. How do we know this? We just subtitute $\frac{3}{5}$ in place of z in the equation of the sphere; the resulting equation is $x^2+y^2+\frac{9}{25} = 1$, or $x^2+y^2 = \frac{16}{25}$.

A **circular paraboloid**, oriented around the z axis, has the equation $z = \frac{x^2+y^2}{r^2}$.

- The trace relative to the plane $z = 0$ is the single point $(0,0,0)$. This is the lowest point on the surface (i.e., there is no point on the surface with a negative z coordinate).
- The trace relative to the plane $z = 1$ is the circle $x^2+y^2 = r^2$, which has radius r.
- The trace relative to the plane $z = z_0$ (where z_0 is positive) is the circle $x^2+y^2 = z_0 r^2$, whose radius is $r\sqrt{z_0}$.
- The trace relative to the plane $y = 0$ is the upward-opening parabola $z = \frac{1}{r^2}x^2$, whose vertex is the point $(x,z) = (0,0)$.
- The trace relative to the plane $x = 0$ is the upward-opening parabola $z = \frac{1}{r^2}y^2$, whose vertex is the point $(y,z) = (0,0)$.
- The trace relative to the plane $y = y_0$ is the upward-opening parabola $z = \frac{1}{r^2}x^2 + \frac{(y_0)^2}{r^2}$, whose vertex is the point $(x,z) = (0,\frac{(y_0)^2}{r^2})$.
- The trace relative to the plane $x = x_0$ is the upward-opening parabola $z = \frac{1}{r^2}y^2 + \frac{(x_0)^2}{r^2}$, whose vertex is the point $(y,z) = (0,\frac{(x_0)^2}{r^2})$.

This surface passes the vertical line test, so we have a function, $f(x,y) = \frac{x^2+y^2}{r^2}$. Its domain is the entire x,y plane, and its range is $z \in [0,\infty)$.

An **elliptic paraboloid** is like a circular parabolid, except it has elliptic cross-sections in place of circular cross-sections. Assuming it is oriented around the z axis, its equation is $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

A **hyperboloid of one sheet**, oriented around the z axis, has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

- For any real number z_0, the trace relative to the plane $z = z_0$ is an ellipse. In particular, when $z = 0$, we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- For any real number x_0, the trace relative to the plane $x = x_0$ is a hyperbola. In particular, when $x = 0$, we get $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
- For any real number y_0, the trace relative to the plane $y = y_0$ is a hyperbola. In particular, when $y = 0$, we get $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$.

This surface does not pass the vertical line test and hence is not a function. However, if we restrict ourselves to the top half of the surface–i.e., to the part lying on or above the x,y plane–then it passes the vertical line test and is a function, $f(x,y) = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}$. The domain is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ together with its *exterior*, i.e., $\frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1$. The range is $z \in [0,\infty)$.

A **hyperboloid of two sheets**, oriented around the z axis, has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$, or $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. This surface consists of two disjoint parts (or "sheets"), one lying above the x,y plane and the other lying below the x,y plane. The lowest point of the upper sheet is the point $(0,0,c)$, and the highest point of the lower sheet is $(0,0,-c)$.

- For any z_0 between $-c$ and c, the trace relative to the plane $z = z_0$ is the empty set.
- For any z_0 that is above c or below $-c$, the trace relative to the plane $z = z_0$ is an ellipse.
- For any real number x_0, the trace relative to the plane $x = x_0$ is a hyperbola. In particular, when $x = 0$, we get $-\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
- For any real number y_0, the trace relative to the plane $y = y_0$ is a hyperbola. In particular, when $y = 0$, we get $-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$.

This surface does not pass the vertical line test and hence is not a function. However, if we restrict ourselves to the top half of the surface–i.e., to the branch lying above the x,y plane–then it passes the vertical line test and is a function, $f(x,y) = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}$. Its domain is the entire x,y plane, and its range is $z \in [c,\infty)$.

A **circular cone**, oriented around the z axis, has the equation $\frac{x^2 + y^2}{r^2} = \frac{z^2}{c^2}$.

- The trace relative to the plane $z = 0$ is the point $(0,0)$.
- For any nonzero z_0, the trace relative to the plane $z = z_0$ is a circle. In particular, when $z = c$ or $z = -c$, we get $x^2 + y^2 = r^2$.

- The trace relative to the plane $x = 0$ is the pair of lines $z = \frac{c}{r}y$ and $z = -\frac{c}{r}y$, which intersect at $(y,z) = (0,0)$.
- The trace relative to the plane $y = 0$ is the pair of lines $z = \frac{c}{r}x$ and $z = -\frac{c}{r}x$, which intersect at $(x,z) = (0,0)$.
- For any nonzero x_0, the trace relative to the plane $x = x_0$ is a hyperbola. In particular, when $x = r$ or $x = -r$, we get $-\frac{y^2}{r^2} + \frac{z^2}{c^2} = 1$.
- For any nonzero y_0, the trace relative to the plane $y = y_0$ is a hyperbola. In particular, when $y = r$ or $y = -r$, we get $-\frac{x^2}{r^2} + \frac{z^2}{c^2} = 1$.

This surface does not pass the vertical line test and hence is not a function. However, if we restrict ourselves to the top half of the surface–i.e., to the part lying on or above the x,y plane–then it passes the vertical line test and is a function, $f(x,y) = \frac{c}{r}\sqrt{x^2+y^2}$. Its domain is the entire x,y plane, and its range is $z \in [0,\infty)$.

An **elliptic cone**, oriented around the z axis, has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$.

- The trace relative to the plane $z = 0$ is the point $(0,0)$.
- For any nonzero z_0, the trace relative to the plane $z = z_0$ is an ellipse. In particular, when $z = c$ or $z = -c$, we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- The trace relative to the plane $x = 0$ is the pair of lines $z = \frac{c}{b}y$ and $z = -\frac{c}{b}y$, which intersect at $(y,z) = (0,0)$.
- The trace relative to the plane $y = 0$ is the pair of lines $z = \frac{c}{a}x$ and $z = -\frac{c}{a}x$, which intersect at $(x,z) = (0,0)$.
- For any nonzero x_0, the trace relative to the plane $x = x_0$ is a hyperbola. In particular, when $x = a$ or $x = -a$, we get $-\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
- For any nonzero y_0, the trace relative to the plane $y = y_0$ is a hyperbola. In particular, when $y = b$ or $y = -b$, we get $-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$.

This surface does not pass the vertical line test and hence is not a function. However, if we restrict ourselves to the top half of the surface–i.e., to the part lying on or above the x,y plane–then it passes the vertical line test and is a function, $f(x,y) = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$. Its domain is the entire x,y plane, and its range is $z \in [0,\infty)$.

A **hyperbolic paraboloid**, oriented around the z axis, has the equation $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$.

- The trace relative to the plane $z = 0$ is the pair of lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, which intersect at $(x,y) = (0,0)$.
- For any nonzero z_0, the trace relative to the plane $z = z_0$ is a hyperbola. In particular, for $z = 1$, we get $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- For any real number x_0, the trace relative to the plane $x = x_0$ is a downward-opening parabola. In particular, for $x = 0$, we get $z = -\frac{1}{b^2}y^2$, which has vertex $(y,z) = (0,0)$. For $x = \pm a$, we get $z = -\frac{1}{b^2}y^2 + 1$, which has vertex $(y,z) = (0,1)$.
- For any real number y_0, the trace relative to the plane $y = y_0$ is an upward-opening parabola. In particular, for $y = 0$, we get $z = \frac{1}{a^2}x^2$, which has vertex $(x,z) = (0,0)$. For $y = \pm b$, we get $z = \frac{1}{a^2}x^2 - 1$, which has vertex $(x,z) = (0,-1)$.

This surface passes the vertical line test and hence is a function, $f(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2}$. Its domain is the entire x,y plane. Its range is $z \in (-\infty, \infty)$.

An alternate version of a hyperbolic paraboloid oriented around the z axis is $z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}$. This is similar to what we considered above, except the downward-opening parabolas and the upward-opening parabolas are reversed.

The term **quadric surfaces** encompasses spheres, ellipsoids, circular and elliptic paraboloids, hyperboloids of one or two sheets, circular and elliptic cones, and hyperbolic paraboloids.

A quadric surface can be *shifted* (or *translated* or *displaced*). For instance, an ellipsoid centered at the point (x_1, y_1, z_1) would have the equation $\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} + \frac{(z-z_1)^2}{c^2} = 1$. On the other hand, the equation $\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2} - \frac{z^2}{c^2} = 1$ represents a hyperboloid of one sheet which is still oriented vertically, but no longer around the z axis. Its new vertical axis would be the line parallel to the z axis and passing through the point $(x_1, y_1, 0)$, which is the line $\mathbf{r}(t) = < x_1, y_1, t >$.

Given an equation of a quadric surface, we may need to complete the square to put the equation into a recognizable form. For example, given the equation $-16x^2 - 4y^2 + z^2 + 64x - 80 = 0$, we rewrite it as $-16(x-2)^2 - 4y^2 + z^2 = 16$. Dividing by -16, we get $(x-2)^2 + \frac{y^2}{4} - \frac{z^2}{16} = -1$, or $\frac{(x-2)^2}{1^2} + \frac{y^2}{2^2} - \frac{z^2}{4^2} = -1$. This is a hyperboloid of two sheets. Its vertical axis is the line parallel to the z axis and passing through the point $(2, 0, 0)$, which is the line $\mathbf{r}(t) = < 2, 0, t >$.