12.1) Two-Dimensional and Three-Dimensional Geometry

The Distance Formula:

- In the x,y plane, the distance between the points (x_1,y_1) and (x_2,y_2) is $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.
- In x,y,z space, the distance between the points (x_1,y_1,z_1) and (x_2,y_2,z_2) is $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

The Midpoint Formula:

- In the x,y plane, the midpoint of the line segment with endpoints (x_1,y_1) and (x_2,y_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.
- In x,y,z space, the midpoint of the line segment with endpoints (x_1,y_1,z_1) and (x_2,y_2,z_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$.

In two-dimensional space, a **circle** is the set of points that are a fixed distance from a given point. This given point is called the **center** of the circle, and the fixed distance is called the **radius**.

In the x,y plane, if the center of a circle is the point (h,k) and the radius is r, then the equation of the circle is $(x-h)^2+(y-k)^2 = r^2$. If the circle is centered at the origin, then its equation simplifies to $x^2+y^2 = r^2$.

The *extreme points* of a circle are the circle's highest, lowest, rightmost, and leftmost points. For the circle $(x-h)^2+(y-k)^2 = r^2$, these points are $(h,k+r)$, $(h,k-r)$, $(h+r,k)$, and $(h-r,k)$. For the circle $x^2+y^2 = r^2$, these points are $(0,r)$, $(0,-r)$, $(r,0)$, and $(-r,0)$, which are also the circle's y and x intercepts.

In the x,z plane, the equation $(x-h)^2+(z-k)^2 = r^2$ represents a circle with radius r centered at (h,k). In the y,z plane, the equation $(y-h)^2+(z-k)^2 = r^2$ represents a circle with radius r centered at (h,k).

A sphere is the three-dimensional version of a circle. In three-dimensional space, a **sphere** is the set of points that are a fixed distance from a given point. This given point is called the **center** of the sphere, and the fixed distance is called the **radius**. If the center is the point (h,k,l) and the radius is r, then the equation of the sphere is $(x-h)^2+(y-k)^2+(z-l)^2 = r^2$. If the sphere is centered at the origin, then its equation simplifies to $x^2+y^2+z^2 = r^2$.

The *extreme points* of the sphere $(x-h)^2+(y-k)^2+(z-l)^2 = r^2$ are $(h,k,l+r)$, $(h,k,l-r)$, $(h,k+r,l)$, $(h,k-r,l)$, $(h+r,k,l)$, and $(h-r,k,l)$. For the sphere $x^2+y^2+z^2 = r^2$, these points are $(0,0,r)$, $(0,0,-r)$, $(0,r,0)$, $(0,-r,0)$, $(r,0,0)$, and $(-r,0,0)$, which are also the sphere's z, y, and x intercepts.

A circle is an example of a **curve**, and a sphere is an example of a **surface**. Later on, we will examine curves in three-dimensional space, but for now we will only consider curves in two-dimensional space (with one exception–we will address vertical lines in three-dimensional space). Surfaces, of course, arise only in three-dimensional space.

A **line** is a special case of a curve–it may be thought of as a curve that is *straight*. A curve that is not a line is a **non-linear curve**. A **plane** is a special case of a surface–it may be thought of as a surface that is *flat*. A surface that is not a plane is a **non-planar surface**.

- In the x,y plane, a *vertical line* is any line perpendicular to the x axis (it may be the y axis or any line parallel to the y axis).
- In the x,z plane, a *vertical line* is any line perpendicular to the x axis (it may be the z axis or any line parallel to the z axis).
- In the y,z plane, a *vertical line* is any line perpendicular to the y axis (it may be the z axis or any line parallel to the z axis).
- In x,y,z space, a *vertical line* is any line perpendicular to the x,y plane (it may be the z axis or any line parallel to the z axis).

In two-dimensional space, a curve is represented by an equation involving two variables. In three-dimensional space, a surface is represented by an equation involving three variables. In both situations, we say the equation defines a **relation** between or among the variables. The curve or surface is the set of all points in the given space satisfying the equation, and is referred to as the **graph** of the relation.

- In the x,y plane, a curve is represented by an equation in x and y, which defines a relation between x and y.
- In the x,z plane, a curve is represented by an equation in x and z, which defines a relation between x and z.
- In the y,z plane, a curve is represented by an equation in y and z, which defines a relation between y and z.
- In x,y,z space, a surface is represented by an equation in x, y, and z, which defines a relation among x, y, and z.

A curve or a surface may or may not represent a **function**.

- In the x,y plane, a curve may or may not represent y as a function of x.
- In the x,z plane, a curve may or may not represent z as a function of x.
- In the y,z plane, a curve may or may not represent z as a function of y.
- In x,y,z space, a surface may or may not represent z as a function of x and y.

It is possible to interchange the roles of the variables–for example, to consider whether a curve represents x as a function of y, or to consider whether a surface represents y as a function of x and z. However, unless otherwise specified, the above frameworks are the ones we will focus on.

In the x,y plane, a curve represents a function if no two points on the curve share a common x coordinate. We may express this idea through the **vertical line test**: If every vertical line intersects the curve at no more than one point, then the curve represents a function, but if any vertical line intersects the curve at multiple points, then the curve does not represent a function.

The vertical line test can also be applied in the x,z plane or in the y,z plane to determine whether z is a function of x or $y,$ respectively.

In x,y,z space, a surface represents a function if no two points on the surface share common x and y coordinates. We may express this idea through the **vertical line test**: If every vertical line intersects the surface at no more than one point, then the surface represents a function, but if any vertical line intersects the surface at multiple points, then the surface does not represent a function.

The circle $x^2 + y^2 = r^2$ fails the vertical line test; for instance, the vertical line $x = 0$ intersects the curve at the points $(0,r)$ and $(0,-r)$. If we solve for y in terms of $x,$ we get $y = \pm\sqrt{r^2 - x^2}$. Because of the "plus-or-minus" sign, this equation will generate two values of y from one value of $x,$ for instance, $y = \pm r$ from $x = 0.$ Hence, y is not a function of $x.$

The sphere $x^2 + y^2 + z^2 = r^2$ fails the vertical line test; for instance, the z axis intersects the surface at the points $(0,0,r)$ and $(0,0,-r)$. If we solve for z in terms of x and $y,$ we get $z = \pm\sqrt{r^2 - x^2 - y^2}$. Because of the "plus-or-minus" sign, this equation will generate two values of z from one pair of values for x and $y,$ for instance, $z = \pm r$ from $(x,y) = (0,0)$. Hence, z is not a function of x and $y.$

Whereas the complete circle $x^2 + y^2 = r^2$ is not a function, if we restrict ourselves to the top half of the circle–i.e., to the part of the circle lying on or above the x axis–we have a **semicircle**, which *is* a function. The equation of the semicircle is $y = \sqrt{r^2 - x^2}$. If we call this function $f,$ then we may write the formula $f(x) = \sqrt{r^2 - x^2}$. The domain of this function is the interval $[-r,r]$ on the x axis. (If x were outside this interval, the radicand would be negative and the formula would generate an imaginary result.) The range of this function is the interval $[0,r]$ on the y axis. This kind of function is known as a **semicircular function**.

Whereas the complete sphere $x^2 + y^2 + z^2 = r^2$ is not a function, if we restrict ourselves to the top half of the sphere–i.e., to the part of the sphere lying on or above the x,y plane–we have a **hemisphere**, which *is* a function. The equation of the hemisphere is $z = \sqrt{r^2 - x^2 - y^2}$. If we call this function $f,$ then we may write the formula $f(x,y) = \sqrt{r^2 - x^2 - y^2}$. The domain of this function is the circular disk $x^2 + y^2 \leq r^2$ in the x,y plane. (A **circular disk** is a circle together with its interior.) (If the point (x,y) were outside this disk, the radicand would be negative and the formula would generate an imaginary result.) The range of this function is the interval $[0,r]$ on the z axis. This kind of function is known as a **hemispherical function**.

Starting with a curve in one of the three coordinate planes (i.e., the x,y plane or the x,z plane or the y,z plane), we convert it into a surface in three-dimensional space, through a process known as **orthogonal projection**. If the original curve is a *line*, then its orthogonal projection is a *plane*. If the original curve is *not* a line, then we refer to its orthogonal projection as a **cylinder**.

- The orthogonal projection of a curve in the x,y plane (i.e., in the plane $z = 0$) is said to be "parallel to the z axis."
- The orthogonal projection of a curve in the x,z plane (i.e., in the plane $y = 0$) is said to be "parallel to the y axis."
- The orthogonal projection of a curve in the y,z plane (i.e., in the plane $x = 0$) is said to be "parallel to the x axis."

The most famous kind of cylinder is a **circular cylinder**. This is the cylinder obtained by orthogonally projecting a *circle*. For instance, if we start with the circle $x^2 + y^2 = r^2$ in the x,y plane and project it orthogonally into three-dimensional space, we obtain a "vertical" circular cylinder whose center is the z axis. It extends infinitely high and infinitely low. On the other hand, if we start with the circle $x^2 + z^2 = r^2$ in the x,z plane and project it orthogonally into three-dimensional space, we obtain a "horizontal" circular cylinder whose center is the y axis. It extends infinitely far to the "left" and to the "right." (I put these directions in quotation marks because left and right are subjective, depending on where you are standing.)

(In a basic geometry class, a "cylinder" is necessarily circular, has finite extent, and is capped off by circular disks at each end, thus forming a closed solid.)

A **parabolic cylinder** is the cylinder obtained by orthogonally projecting a *parabola*. For instance, if we start with the parabola $y = x^2$ in the x,y plane and project it orthogonally into three-dimensional space, we obtain a "vertical" parabolic cylinder, which runs along the z axis (i.e., every point on the z axis is part of the cylinder). On the other hand, if we start with the parabola $z = x^2$ in the x,z plane and project it orthogonally into three-dimensional space, we obtain a "horizontal" parabolic cylinder, which runs along the y axis (i.e., every point on the y axis is part of the cylinder).

A vertical cylinder is not the graph of a function (i.e., z is not a function of x and y) because it will obviously fail the vertical line test. On the other hand, a horizontal cylinder may be a function. The horizontal circular cylinder $x^2 + z^2 = r^2$ fails the vertical line test and so is not a function; however, if we take only the top half (i.e., the part lying on or above the x,y plane), it passes the vertical line test, and we have the function $f(x,y) = \sqrt{r^2 - x^2}$, where the domain is $\{(x,y) \mid x \in [-r,r],\ y \in (-\infty,\infty)\}$ and the range is $z \in [0,r]$. The horizontal parabolic cylinder $z = x^2$ passes the vertical line test, so we have the function $f(x,y) = x^2$, where the domain is the entire x,y plane and the range is $z \in [0,\infty)$.

So far, we have discussed orthogonally projecting points of a plane into three dimensional space. Conversely, we can orthogonally project a set of points in three-dimensional space (e.g., a surface) onto a plane (typically the x,y plane, i.e., the plane $z = 0$). To project a set of points onto the plane $z = 0$, we simply replace each point's z coordinate with 0. For example, the point $(-3,7,19)$ would project to the point $(-3,7,0)$.

This process is analogous to a process with which you are already familiar–namely, orthogonally projecting a set of points in two-dimensional space (e.g., a curve) onto a line (either the x axis, which is the line $y = 0$, or the y axis, which is the line $x = 0$). You use this process to determine the domain and range of a relation from its graph. For example, consider the circle $(x-5)^2 + (y-4)^2 = 9$, which has center $(5,4)$ and radius 3. Its extreme points are $(5,7)$, $(5,1)$, $(8,4)$, and $(2,4)$. If we project the circle onto the x axis, we obtain the interval $[2,8]$, which is the domain of the relation. If we project the circle onto the y axis, we obtain the interval $[1,7]$, which is the range of the relation.

If we project the sphere $x^2 + y^2 + z^2 = 16$ onto the x,y plane (i.e., the plane $z = 0$), we obtain the circular disk $x^2 + y^2 \le 16$, which is the domain of the relation. (This is the disk with center $(0,0)$ and radius $4.$)

If we project the sphere $(x-4)^2 + (y-5)^2 + (z-3)^2 = 100$ onto the x,y plane (i.e., the plane $z = 0$), we obtain the circular disk $(x-4)^2 + (y-5)^2 \le 100$, which is the domain of the relation. (This is the disk with center $(4,5)$ and radius $10.$)