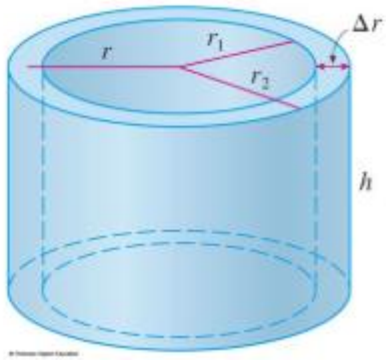
6.3 Volumes by Cylindrical Shells

In this section we will discuss the method used to find the volume of an object when it is difficult to find the inner and outer radius: The Method of Cylindrical Shells.

Consider the figure below:



The volume **V** is calculated by subtracting the volume **V**$_i$, the volume of the inner cylinder from **V**$_o$, the volume of the outer cylinder: Let **V**$_i$ **=** **V**$_1$ **,** the volume of the inner cylinder and **V**$_o$ **=** **V**$_2$, the volume of the outer cylinder, then we have:

$$V = V_2 - V_1$$

$$= \pi r_2^2 h - \pi r_1^2 h = \pi h(r_2^2 - r_1^2) = \pi h(r_2 + r_1)(r_2 - r_1)$$

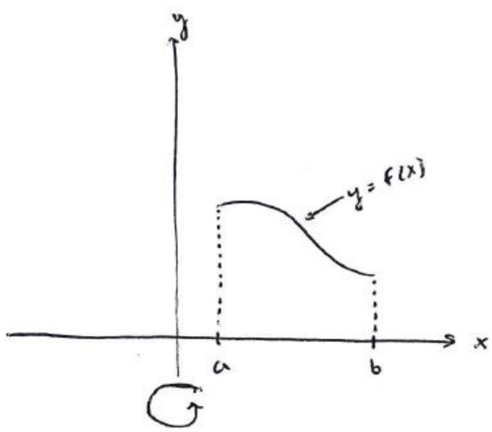
Now – using a little algebra we can rewrite this as: $V = 2\pi\left(\frac{r_2+r_1}{2}\right)h(r_2 - r_1)$

If $r = \frac{1}{2}(r_2 + r_1)$ and $\Delta x = r_2 - r_1$, then the formula for the volume of a cylindrical shell becomes

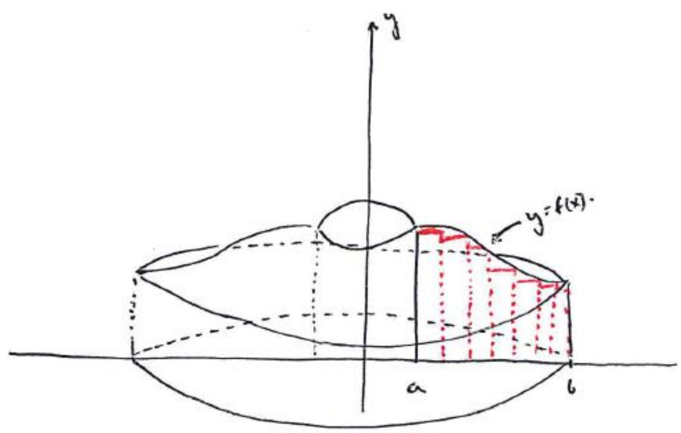
$$\boldsymbol{V = 2\pi r h \Delta x}$$

$$V = [circumference][height][thickness]$$

Now consider a region that is a bit more difficult.



rotate about y-axis



We divide the interval [a, b] into **n** subintervals $[x_{i-1}, x_i]$ of equal width, Δx, and let $\bar{x}_i$ be the midpoint of the $\boldsymbol{i^{th}}$ subinterval. The height of the rectangle is $\boldsymbol{f(\bar{x}_i)}$**.** This gives us a formula for the volume of each rectangle:

$$V_i = 2\pi \cdot \bar{x}_i \cdot f(\bar{x}_i)\Delta x.$$

Adding all of the rectangles from **x = a** to **x = b** give the total volume to be

$$V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi \cdot \bar{x}_i \cdot f(\bar{x}_i)\Delta x$$

Again the approximation is more accurate as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \cdot \bar{x}_i \cdot f(\bar{x}_i)\Delta x = \int_a^b 2\pi \cdot x \cdot f(x)dx$$

The **volume** of the solid, obtained by rotating about the **y-axis** the region under the curve **y = f(x)** from **x=a** to **x=b,** is:

$$V = \int_a^b 2\pi \cdot x \cdot f(x)dx \quad where\ 0 \le a \le b$$

Example: Find the volume of the solid obtained by rotating about the y-asix the region bounded by $y = x^2 - x^3$ and $y = 0$. (Note: To find the limits of integration set **f(x) = 0** and solve for **x**.)

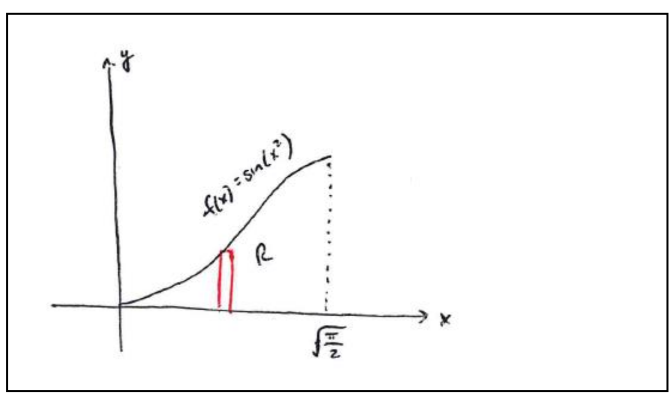
$$V = \int_a^b [circumference][height][thickness] = \int_0^1 [2\pi x][x^2 - x^3][dx]$$

$$= 2\pi \int_0^1 (x^3 - x^4)dx = 2\pi \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 2\pi\left(\frac{1}{4} - \frac{1}{5}\right) = 2\pi\frac{1}{20} = \frac{\pi}{10}$$

Now suppose that the region is bounded between two curves **y = f(x)** and **y = g(x)** on [a, b]. If **R** is the region bounded by the curves **y = f(x)** and **y = g(x)** between the lines **x = a** and **x = b**, the volume of the solid generated when **R** is revolved about the **y-axis** is

$$V = \int_a^b 2\pi x(f(x) - g(x))dx$$

Example: Let **R** be the region bounded by the graph of $f(x) = sin\,(x^2)$, the **x-axis**, and the vertical line $x = \sqrt{\frac{\pi}{2}}$. Find the volume of the solid generated by rotating about the **y-axis.**



First graph the information:

The **x-axis** can be written as **g(x) = 0,** therefore

The volume of each rectangle $= 2\pi x(f(x) - g(x))\Delta x$

$V = 2\pi x(sin(x^2) - 0)\Delta x.$

The radius = **x** and the height = **f(x)**

$$V = \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \cdot \sin(x^2)dx$$

Let $\mathbf{u = x^2} \Rightarrow \mathbf{du = 2xdx}$. When $\mathbf{x = 0, u = 0}$ and when $\mathbf{x = \sqrt{\frac{\pi}{2}}, u = \frac{\pi}{2}}$ so the limits of integration are 0 and $\frac{\pi}{2}$.

$$\int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \cdot \sin(x^2)dx = \int_0^{\frac{\pi}{2}} \pi \cdot \sin(u)du = \pi \int_0^{\frac{\pi}{2}} \sin(u)du$$

$$\pi[-\cos(u)]_0^{\pi/2} = \pi(-cos(\pi/2) - (-cos(0)) = \boldsymbol{\pi}$$

Example: Let **R** be the region in the first quadrant bounded by the graph of $y = \sqrt{x-2}$ and $y = 2$. Find the volume of a solid generated when revolved about the **x-axis.**

Notice we are revolving about the **x-axis** instead of the **y-axis**, therefore the integration will be with respect to y. Meaning, y is now the independent variable and the functions will need to be rewritten in terms of **y**. The limits of integration are **y = 0** and **y = 2.** (The lower limit of integration comes from the smallest value of the domain of the boundary $y = \sqrt{x-2}$ - when x = 2, y = 0. The upper limit of integration comes from the other boundary $y = 2$.)

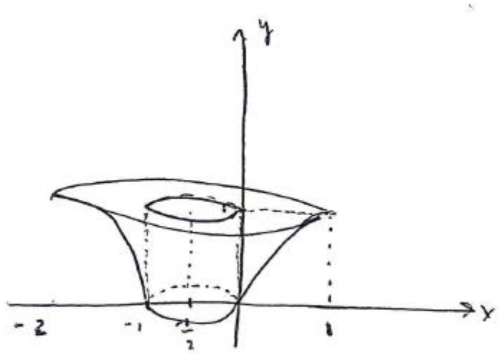
Solve $\boldsymbol{y = \sqrt{x-2}}$ for x and you will get $\boldsymbol{x = y^2 + 2}$. Let $p(y) = y^2 + 2$ and $q(y) = 0$ (since it is in the 1st quadrant, it is also bounded by **x = 0** which is written as $q(y) = 0$)

$$V = \int_0^2 2\pi y(p(y) - q(y))dy = \int_0^2 2\pi y(y^2 + 2)dy = 2\pi \int_0^2 (y^3 + 2y)dy$$

$$= 2\pi\left[\frac{y^4}{4} - y^2\right]_0^2 = 2\pi\left[\left(\frac{2^4}{4} + 4\right) - 0\right] = 2\pi[8] = \mathbf{16\boldsymbol{\pi}}$$

Example: Let **R** be the region bounded by the curve $y = \sqrt{x}$, $y = 1$ and the $y - axis$. Find the volume of the solid generated when **R** is revolved about the line $x = -\frac{1}{2}$.

The graph would look like:



The radius is $\sqrt{x} + \frac{1}{2}$, the height is $1 - \sqrt{x}$ and the limits of integration are 0 and 1.

$$\int_0^1 2\pi\left(\sqrt{x}+\frac{1}{2}\right)\left(1-\sqrt{x}\right)dx \;=\; 2\pi\int_0^1\left(\frac{1}{2}x^{\frac{1}{2}}-x+\frac{1}{2}\right)dx$$

$$2\pi\left[\frac{1}{2}\cdot\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{2}x^{\frac{1}{2}}+\frac{1}{2}x\right]_0^1 \;=\; 2\pi\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{2}\right] \;=\; \boldsymbol{\frac{2\pi}{3}}$$