4.3 How Derivatives Affect the Shape of a Graph

Remember that the derivative of a function ***f*** tells us the direction in which the curve proceeds at each point. The derivative of *f(x)*, ***f'(x)***, tells us if the function is increasing or decreasing at every ***x*** in the domain of ***f(x)***. If the derivative of ***f*** is positive in a certain interval then the function is increasing, if the derivative of ***f*** is negative in a certain interval then the function is decreasing. From this information we have developed the following test.

Increasing/Decreasing Test:

a) If ***f'(x)* > 0** on an interval, then ***f*** is increasing on that interval.
b) If ***f'(x)* < 0** on an interval, then ***f*** is decreasing on that interval.

Example: Find the intervals on which ***f*** is increasing and/or decreasing. ***f(x) = sin(x) + cos(x)*** $[0, 2\pi]$

We can use the I/D test but first we must find where ***f'(x)* = 0**.

$\cos x - \sin x = 0$

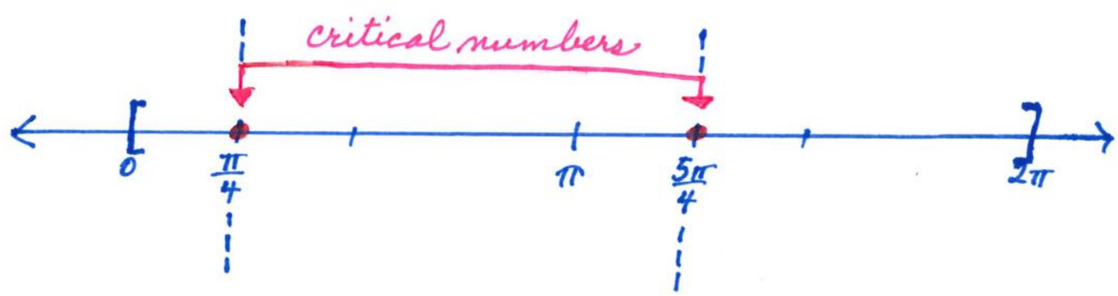
$\cos x = \sin x$

$1 = \frac{\sin x}{\cos x}$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$

These two values are the **critical numbers** of ***f*** and they divide the domain into three intervals, $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right)$ See the sign graph below.



Within each interval, we test ***f'(x)*** and determine if it is positive or negative.

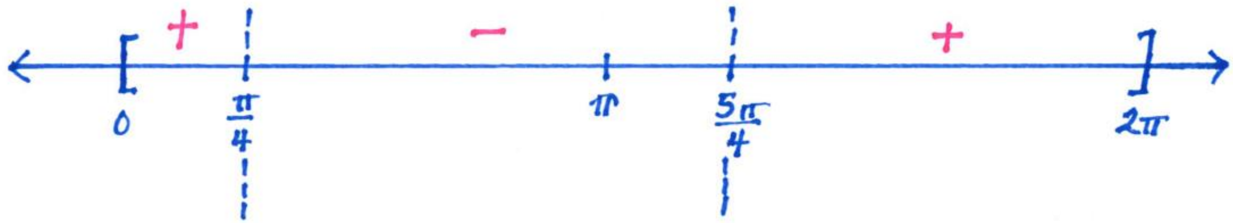
We will now plug in values for ***x*** within each interval into ***f'(x)***. Although we can choose any value in the interval, the values I have chosen are $\boldsymbol{x = \frac{\pi}{6}, \pi, \frac{3\pi}{2}}$

$f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2} > 0$ ∴ the function is **increasing** on the interval $\left(0, \frac{\pi}{4}\right)$.

$f'(\pi) = \cos\pi - \sin\pi = -1 - 0 = -1 < 0$ ∴ the function is **decreasing** on the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

$f'\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} - \sin\frac{3\pi}{2} = 0 - (-1) = 1 > 0$ ∴ the function is **increasing** on the interval $\left(\frac{5\pi}{4}, 2\pi\right)$.

The results are marked on the sign graph below.



From section 4.1 we saw that if f has a local maximum or a local minimum at c, then c must be a critical number of f, but not every critical number gives us a local maximum or minimum. For this reason we need a **test** that will tell us whether or not f has a local maximum or minimum at a critical number.

The First Derivative Test: Suppose c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local **maximum** at c.

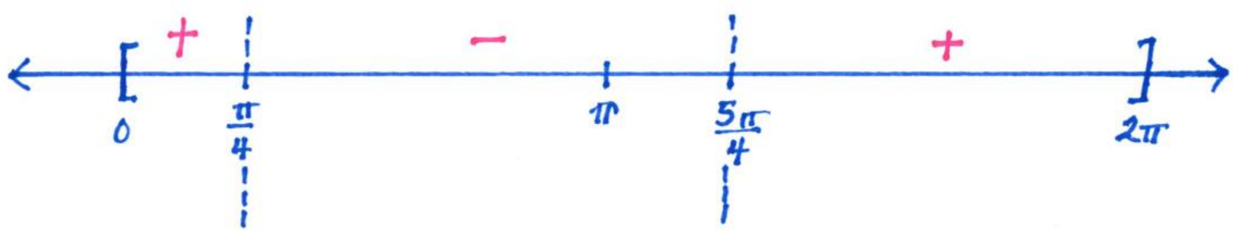
b) If f' changes from negative to positive at c, then f has a local **minimum** at c.

c) If f' is positive to the left and right of c, or negative to the left and right of c, (in other words, the sign does not change at c) then f has **no** maximum or minimum at c.

Example: Find the local minimum and/or maximum of the function in the last example.

$f(x) = sin(x) + cos(x)$ $[0, 2\pi]$

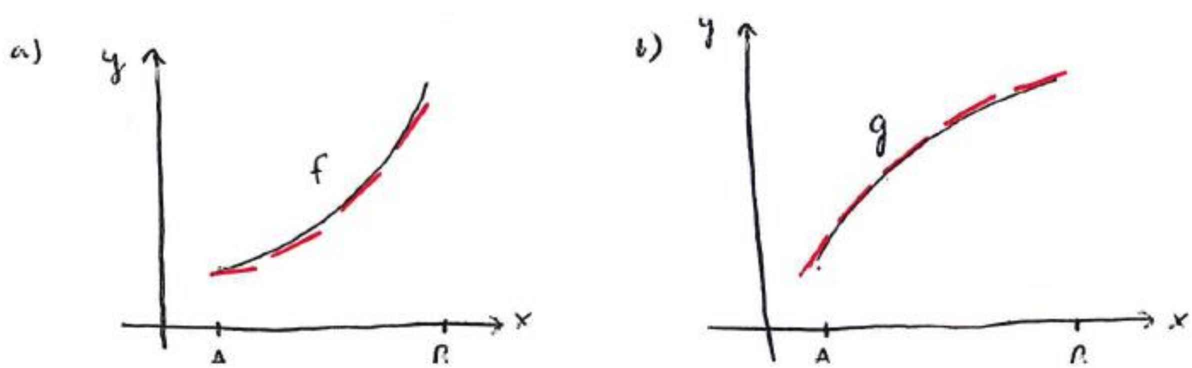
Using the First Derivative Test and looking at the sign graph below –



We can see that there is a **maximum** at $\boldsymbol{x = \frac{\pi}{4}}$ and a **minimum** at $\boldsymbol{x = \frac{5\pi}{4}}$. Therefore, the local maximum is $x = \frac{\pi}{4}$ and the local minimum is $x = \frac{5\pi}{4}$.

Now that we have a test to analyze if a maximum or minimum occurs, it would be nice to see how the function is behaving between these x-values.

Notice that graph a) and graph b) both join point A to point B but they bend in different directions.



In graph a), the curve lies above the tangents. This means the function is concave upward on the interval (A, B) In graph b) the curve lies below the tangets. This means the function is concave downward on the interval (A, B).

Definition: If the graph of f lies above all of its tangents on an interval I, then f is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

From graph a) we can see that, going left to right, the slopes of the tangents are increasing. This means that the derivative f' is an increasing function and therefore its derivative f'' is positive. Similarly, in graph b) the slopes of the tangents are decreasing from left to right, so f' is decreasing and therefore f'' is negative. This is called the **Concavity Test.**

Concavity Test:

a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition: A point P on a curve $y = f(x)$ is called an ***inflection point*** if f is continuous at P and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

In view of the Concavity Test, there is a point of inflection at any point where the second derivative changes sign.

Example: Find the **intervals of concavity** and the **inflection points** for $f(x) = \sin(x) + \cos(x)$ $[0, 2\pi]$

From the previous examples we have seen that $f(x)$ has a max at $x = \frac{\pi}{4}$ and a min at $x = \frac{5\pi}{4}$. We have also determined that the function is increasing on the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and decreasing on the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. To find concavity we have to use the Second Derivative Test and find the x–values that make $f''(x) = 0$.

$f'(x) = \cos(x) - \sin(x)$

$f''(x) = -\sin(x) - \cos(x) = 0$

$-\sin(x) = \cos(x)$

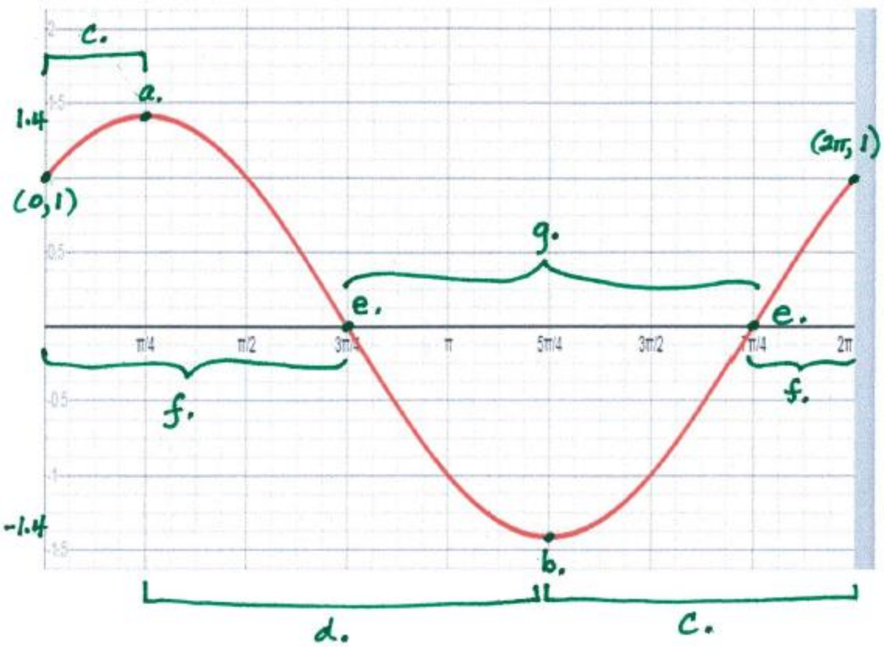
$\tan(x) = -1$ thus $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ are the values of x where the inflection points occur. Notice that this is where the function changes concavity.

We can divide the real number line from $0 \leq x \leq 2\pi$ into intervals with these numbers as endpoints and create the following chard to find the intervals of concavity.

Interval	f''(x)=-sin(x)-cos(x)	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$-$	downward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$+$	upward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$-$	downward

The function is concave up in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ and concave down in $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$.

Example: Using all of the information found for $f(x) = \sin x + \cos x$ in $[0, 2\pi]$, plot the graph.



a. Local max at $x = \frac{\pi}{4}, y \approx 1.4$

b. Local min at $x = \frac{5\pi}{4}, y \approx -1.4$

c. Increasing: $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$

d. Decreasing: $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

e. Points of Inflection: $\left(\frac{3\pi}{4}, 0\right)$ & $\left(\frac{7\pi}{4}, 0\right)$

f. Concave down: $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$

g. Concave up: $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$

$$\begin{aligned} f(0) &= \sin(0) + \cos(0) \quad (0,1) \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f(2\pi) &= \sin(2\pi) + \cos(2\pi) \quad (2\pi, 1) \\ &= 0 + 1 \\ &= 1 \end{aligned}$$