5.2 The Definite Integral

From the last section we know . . .
$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x]$ is found when computing an area.

This same type of limit occurs in a wide variety of situation even when ***f*** is not at positive function.

Definition of a Definite Integral: If ***f*** is a function defined for $\boldsymbol{a \le x \le b}$, we divide the interval [a, b] into **n** subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, x_3, \cdots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, x_3^*, \cdots, x_n^*$ be any sample points (right, left, midpoints, etc...) in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then definite integral of ***f*** from **a** to **b** is

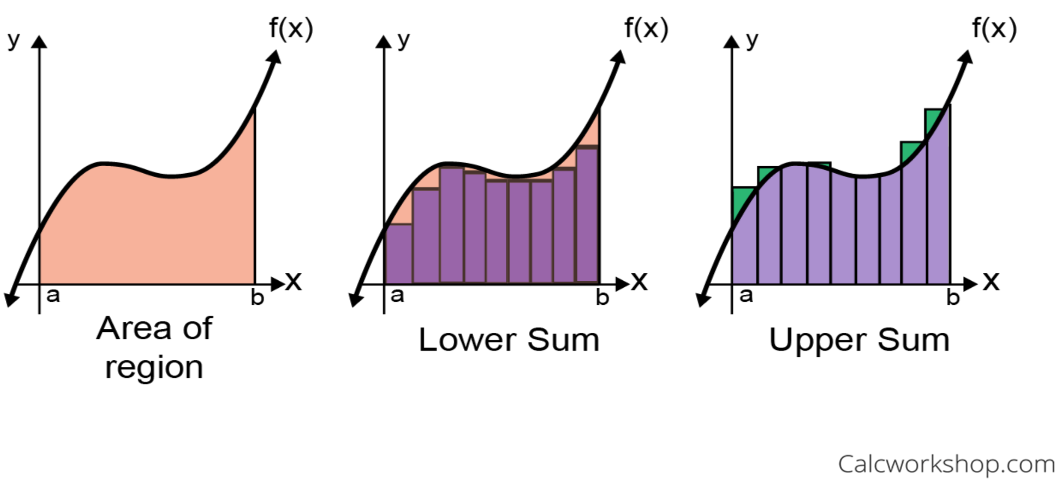
$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$\cdots$ provided that this limit exists and gives the same value for all possible choices of sample points. If it **does** exist, we say that ***f*** is integrable on [a, b].

$\int$ *is the integral sign*. In $\int_a^b f(x)dx$, $\boldsymbol{f(x)}$ is the **integrand, a** and **b** are the limits of integration where **a** is the lower limit and **b** is the upper limit, **dx** indicates that the independent variable is **x** and the function is being integrated with respect to **x.**

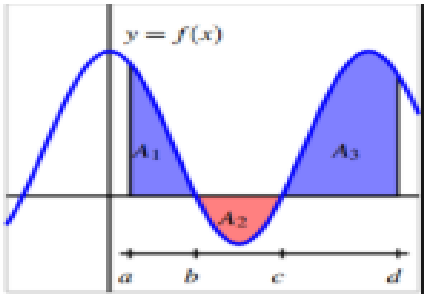
$\sum_{i=1}^{n} f(x_i^*)\Delta x$ is the Riemann Sum. If ***f*** is positive, then the Riemann Sum is the sum of area of approximating rectangles.

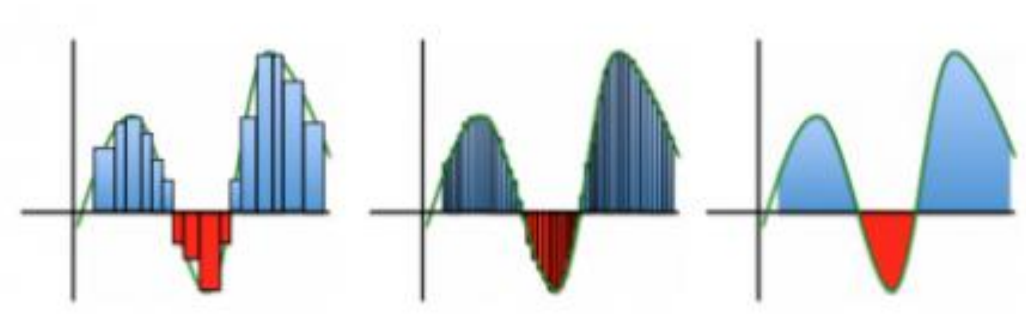
Since $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$, we can see that $\int_a^b f(x)\,dx$ can be interprested as the area under the curve ***y = f(x)*** from **a** to **b.**



If f is both positive and negative, then the Riemann Sum is the sum of the areas of the rectangles that lie above the **x-axis** minus the areas of the rectangles that lie below the **x-axis.** For the diagrams below, the area would be interpreted as:

$$\int_a^b f(x)dx = A_1 + A_3 - A_2$$





We have defined the integral for an integrable function, but not all functions are integrable.

Theorem: If f is integrable on **[a, b]**, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Example: Evaluate $\int_0^3 (x^3 - 6x)dx$ using the second theorem above.

We have $\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$, where $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{\mathbf{3}}{\boldsymbol{n}}$ and $x_i = 0 + i\frac{3}{n} = \frac{\mathbf{3}\boldsymbol{i}}{\boldsymbol{n}}$

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{3i}{n}\right)^2 - 6\left(\frac{3i}{n}\right)\right)\frac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)\frac{3}{n}$$

$$\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left(\frac{27i^3}{n^3} - \frac{18i}{n}\right) = \lim_{n\to\infty} \frac{3}{n}\left[\frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty} \left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right] =$$

$$\lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right] = \lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n^2(n^2+2n+1)}{4}\right) - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right] =$$

$$\lim_{n\to\infty} \left[\frac{81}{4}\left(\frac{\cancel{n^2}(n^2+2n+1)}{\cancel{n^4}\, n^2}\right) - \frac{54}{2}\left(\frac{\cancel{n}(n+1)}{\cancel{n^2}\, n}\right)\right] = \lim_{n\to\infty} \left[\frac{81}{4}\left(\frac{n^2+2n+1}{n^2}\right) - \frac{54}{2}\left(\frac{n+1}{n}\right)\right]$$

$$\lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)-\frac{54}{2}\left(1+\frac{1}{n}\right)\right]=\frac{81}{4}-27=\frac{-27}{4}=-6.25$$

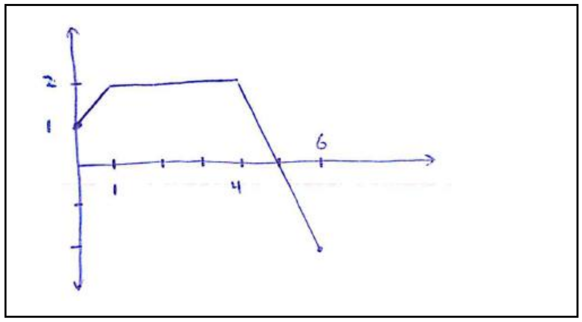
This integral can't be interpreted as an area since ***f*** takes on both positive and negative values and areas can't be negative.

Example: Evaluate the following integrals by interpreting each in terms of geometric areas. In other words, you will use geometric area formulas. f(x) is graphed below.

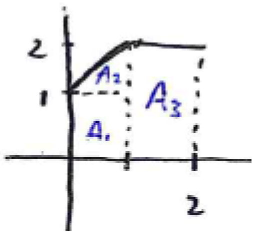
a) $\int_0^2 f(x)dx$

b) $\int_0^4 f(x)dx$

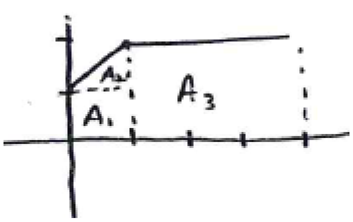
c) $\int_0^6 f(x)dx$



a) $\int_0^2 f(x)dx = A_1 + A_2 + A_3 = (1)(1) + \frac{1}{2}(1)(1) + (2)(1) = 1 + .5 + 2 = \mathbf{3.5}$

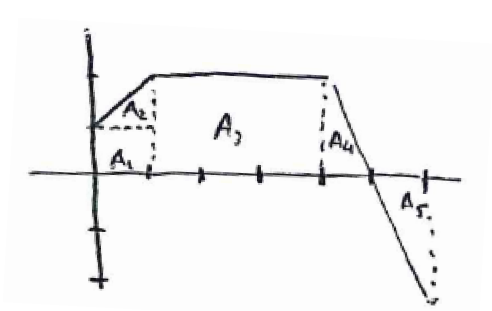


b) $\int_0^4 f(x)dx = A_1 + A_2 + A_3 = (1)(1) + \frac{1}{2}(1)(1) + (3)(2) = 1 + .5 + 6 = \mathbf{7.5}$



c) $\int_0^6 f(x)dx = A_1 + A_2 + A_3 + A_4 - A_5 = (1)(1) + \frac{1}{2}(1)(1) + (3)(2) + \frac{1}{2}(1)(2) - \frac{1}{2}(1)(2)$

$= 1 + .5 + 6 + 1 - 1 = \mathbf{7.5}$



Properties of the Definite Integral – assume that **a < b**.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

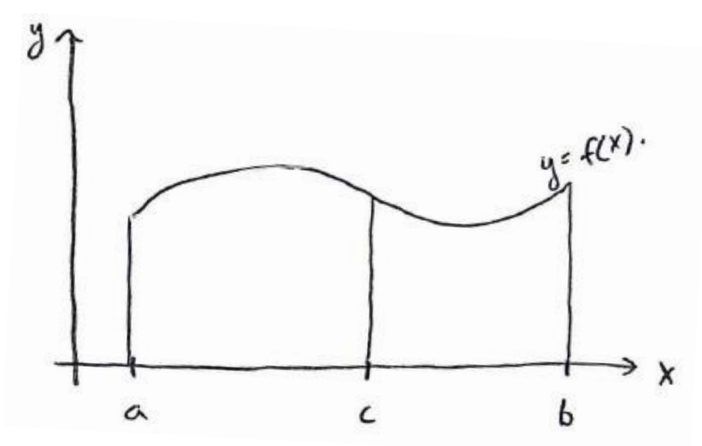
If $\mathbf{a = b}$, then $\Delta x = 0$ and

$$\int_a^a f(x)dx = 0$$

Properties of the Integral

1. $$\int_a^b c \cdot dx = c(b - a), \quad \textit{where } c \textit{ is any constant}$$
2. $$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$
3. $$\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx, \textit{where } c \textit{ is any constant}$$
4. $$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$
5. $$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

For Property #5, below is a graphical representation.



Comparison Properties of the Integral.

6. If $f(x) \geq 0$ for $a \leq x \leq b$, $then$ …

$$\int_a^b f(x)dx \geq 0$$

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, $then$ …

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

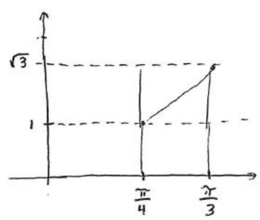
8. If $m \leq f(x) \leq M$ *for* $a \leq x \leq b$, *then* ...

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

All of these properties can be proved.

Example: Use property **8** to estimate the value of the integral.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan(x)\, dx$$

Graph the tangent function from $\frac{\pi}{4}$ to $\frac{\pi}{3}$. 

We see that $1 \leq \tan(x) \leq \sqrt{3}$ for $\frac{\pi}{4} \leq x \leq \frac{\pi}{3}$

Thus:

$$1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x\, dx \leq \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\frac{\pi}{12} \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x\, dx \leq \frac{\sqrt{3}\pi}{12}$$