16.3) The Fundamental Theorem for Line Integrals

As discussed in Section 16.1, a **conservative vector field** is a vector field for which there exists a **potential function**, i.e., a real-valued function whose gradient is the given vector field. The potential function and the vector field have the same domain.

If a vector field $\mathbf{F}$ is conservative, then a potential function for $\mathbf{F}$ may be denoted f.

In two dimensions, we have $\mathbf{F}(x,y) = < p(x,y), q(x,y) >$. If $f(x,y)$ is a potential function for $\mathbf{F}$, then $\nabla f(x,y) = \mathbf{F}(x,y)$. Since $\nabla f(x,y) = < f_x(x,y), f_y(x,y) >$, we must have $f_x(x,y) = p(x,y)$ and $f_y(x,y) = q(x,y)$.

In three dimensions, we have $\mathbf{F}(x,y,z) = < p(x,y,z), q(x,y,z), r(x,y,z) >$. If $f(x,y,z)$ is a potential function for $\mathbf{F}$, then $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$. Since $\nabla f(x,y,z) =$ $< f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) >$, we must have $f_x(x,y,z) = p(x,y,z)$, $f_y(x,y,z) = q(x,y,z)$, and $f_z(x,y,z) = r(x,y,z)$.

In Section 16.1, we cited (but did not answer) two key questions:

1. Given a vector field, how do we decide whether or not it is conservative–i.e., whether or not it *has* a potential function?
2. If a vector field *is* conservative, how do we *find* a potential function for it?

We will now answer both of these questions.

Two-dimensional vector fields:

If f is a potential function for $\mathbf{F}$, then $f_x = p$ and $f_y = q$. It follows that:

- $f_{xx} = p_x$
- $f_{xy} = p_y$
- $f_{yx} = q_x$
- $f_{yy} = q_y$

By Clairaut's Theorem, $f_{xy} = f_{yx}$, so $p_y = q_x$. This is, in fact, our *test* for whether or not a two-dimensional vector field is conservative. (Notice that we can disregard the "pure" second-order partial derivatives, f_{xx} and f_{yy}. In other words, we can disregard p_x and q_y. We need only concern ourselves with the "mixed" second-order partial derivatives, f_{xy} and f_{yx}, which correspond to p_y and q_x.)

In Section 16.1, we mentioned that the vector field $\mathbf{F}(x,y) = < 10x, 6y^2 >$ is conservative. We may now confirm this by noting that $p_y = q_x = 0$.

In Section 16.2, the vector field in Example 1 was $\mathbf{F}(x,y) = < y - x,\ x >$, and the vector field in Examples 2 and 3 was $\mathbf{F}(x,y) = < x + y, x - y >$. Both of these vector fields are conservative; in both cases, we have $p_y = q_x = 1$.

Now let's explore the technique for *finding* the potential function, via *anti-differentiation*.

Since $f_x = p(x,y)$, we can recover f by anti-differentiating $p(x,y)$ with respect to x.

$f = \int f_x \, dx = \int p(x,y) \, dx$. This will give us a sum of at least two terms. Every term besides the last term will be a function of x or a function of both x and y. The last term will be an unknown function of y, denoted $g(y)$.

Gather together the terms containing x (either x alone or both x and y), and call this $\lambda(x,y)$. So $f = \lambda(x,y) + g(y)$.

Differentiating with respect to y, we get $f_y = \lambda_y(x,y) + g'(y)$. Set this equal to $q(x,y)$. So $g'(y) = q(x,y) - \lambda_y(x,y)$. In most cases, some terms will cancel out, giving us a relatively simple formula for $g'(y)$.

Now that we know $g'(y)$, we can recover $g(y)$ by anti-differentiating $g'(y)$ with respect to y, i.e., we find $\int g'(y) \, dy$. When doing so, we typically omit the arbitrary constant term.

Once we have found $g(y)$, we add it to $\lambda(x,y)$. This gives us our formula for f, i.e., $f = \lambda(x,y) + g(y)$.

Example 1: Let $\mathbf{F}(x,y) = < 2xe^{3y} + 3x^2,\ 3x^2e^{3y} + 4y^3 >$. Here, $p(x,y) = 2xe^{3y} + 3x^2$, and $q(x,y) = 3x^2e^{3y} + 4y^3$.

$p_y = q_x = 6xe^{3y}$, so the vector field is conservative.

$f = \int p(x,y) \, dx = \int (2xe^{3y} + 3x^2) \, dx = x^2e^{3y} + x^3 + g(y)$. Here, $\lambda(x,y) = x^2e^{3y} + x^3$.

$f_y = 3x^2e^{3y} + g'(y)$. Here, $\lambda_y(x,y) = 3x^2e^{3y}$.

$3x^2e^{3y} + g'(y) = 3x^2e^{3y} + 4y^3$, so $g'(y) = 4y^3$.

$g(y) = \int g'(y) \, dy = \int 4y^3 dy = y^4$.

So $f = x^2e^{3y} + x^3 + y^4$.

Example 2: Let $\mathbf{F}(x,y) = < x + y, x - y >$, as in Examples 2 and 3 of Section 16.2. Here, $p(x,y) = x + y$, and $q(x,y) = x - y$.

$f = \int p(x,y) \, dx = \int (x + y) \, dx = \frac{1}{2}x^2 + xy + g(y)$. Here, $\lambda(x,y) = \frac{1}{2}x^2 + xy$.

$f_y = x + g'(y)$. Here, $\lambda_y(x,y) = x$.

$x + g'(y) = x - y$, so $g'(y) = -y$.

$g(y) = \int g'(y)\, dy = \int -y\, dy = -\frac{1}{2}y^2$.

So $f = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$.

Three-dimensional vector fields:

If f is a potential function for $\mathbf{F}$, then $f_x = p$, $f_y = q$, and $f_z = r$. It follows that:

- $f_{xy} = p_y$
- $f_{xz} = p_z$
- $f_{yx} = q_x$
- $f_{yz} = q_z$
- $f_{zx} = r_x$
- $f_{zy} = r_y$

(We may ignore the three "pure" second-order partial derivatives, f_{xx}, f_{yy}, and f_{zz}.)

By Clairaut's Theorem, $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, and $f_{yz} = f_{zy}$. Hence:

- $p_y = q_x$
- $p_z = r_x$
- $q_z = r_y$

This is, in fact, our *test* for whether or not a three-dimensional vector field is conservative (i.e., *all three* equations must hold).

In Section 16.1, we mentioned that the vector fields $\mathbf{F}(x,y,z) = < yz, xz, xy >$ and $\mathbf{F}(x,y,z) = < 2xy^3 + 3x^2z^5, 3x^2y^2 + 2yz^4, 4y^2z^3 + 5x^3z^4 >$ are conservative. We may now confirm this.

- In the former case, $p_y = q_x = z$, $p_z = r_x = y$, and $q_z = r_y = x$.
- In the latter case, $p_y = q_x = 6xy^2$, $p_z = r_x = 15x^2z^4$, and $q_z = r_y = 8yz^3$.

In Section 16.2, the vector field in Example 4 was $\mathbf{F}(x,y,z) = < z, x, -y >$. This vector field is *not* conservative, because $p_y = 0$, whereas $q_x = 1$.

Now let's explore the technique for *finding* the potential function, via *anti-differentiation*.

Since $f_x = p(x,y,z)$, we can recover f by anti-differentiating $p(x,y,z)$ with respect to x.

$f = \int f_x\, dx = \int p(x,y,z)\, dx$. This will give us a sum of at least two terms. Every term besides the last term will contain x (possibly accompanied by y and/or z). The last term will be an unknown function of y and z, denoted $g(y,z)$.

Gather together the terms containing x, and call this $\lambda(x,y,z)$. So $f = \lambda(x,y,z) + g(y,z)$.

Differentiating with respect to y, we get $f_y = \lambda_y(x,y,z) + g_y(y,z)$. Set this equal to $q(x,y,z)$. So $g_y(y,z) = q(x,y,z) - \lambda_y(x,y,z)$. In most cases, some terms will cancel out, giving us a relatively simple formula for $g_y(y,z)$.

Now that we know $g_y(y,z)$, we can recover $g(y,z)$ by anti-differentiating $g_y(y,z)$ with respect to y, i.e., we find $\int g_y(y,z)\ dy$. This will give us a sum of at least two terms. Every term besides the last term will be a function of y or a function of both y and z. The last term will be an unknown function of z, denoted $h(z)$.

Gather together the terms containing y (either y alone or both y and z), and call this $\mu(y,z)$. So $g(y,z) = \mu(y,z) + h(z)$.

By substitution, $f = \lambda(x,y,z) + \mu(y,z) + h(z)$.

Differentiating with respect to z, we get $f_z = \lambda_z(x,y,z) + \mu_z(y,z) + h'(z)$. Set this equal to $r(x,y,z)$. So $h'(z) = r(x,y,z) - \lambda_z(x,y,z) - \mu_z(y,z)$. In most cases, some terms will cancel out, giving us a relatively simple formula for $h'(z)$.

Now that we know $h'(z)$, we can recover $h(z)$ by anti-differentiating $h'(z)$ with respect to z, i.e., we find $\int h'(z)\ dz$. When doing so, we typically omit the arbitrary constant term.

Once we have found $h(z)$, we add it to $\lambda(x,y,z) + \mu(y,z)$. This gives us our formula for f, i.e., $f = \lambda(x,y,z) + \mu(y,z) + h(z)$.

Example 3:

Let $\mathbf{F}(x,y) = < 2xy - z^2,\ x^2 + 2z,\ 2y - 2xz >$. Here, $p(x,y,z) = 2xy - z^2$, $q(x,y,z) = x^2 + 2z$, and $r(x,y,z) = 2y - 2xz$.

We confirm the vector field is conservative as follows:

1. $p_y = q_x = 2x$
2. $p_z = r_x = -2z$
3. $q_z = r_y = 2$

$f = \int p(x,y,z)\ dx = \int (2xy - z^2)dx = x^2y - xz^2 + g(y,z)$. Here, $\lambda(x,y,z) = x^2y - xz^2$.

$f_y = x^2 + g_y(y,z)$. Here, $\lambda_y(x,y,z) = x^2$.

$x^2 + g_y(y,z) = x^2 + 2z$, so $g_y(y,z) = 2z$.

$g(y,z) = \int g_y(y,z)\ dy = \int 2z\ dy = 2yz + h(z)$. Here, $\mu(y,z) = 2yz$.

By substitution, $f = \lambda(x,y,z) + \mu(y,z) + h(z) = x^2y - xz^2 + 2yz + h(z)$.

$f_z = -2xz + 2y + h'(z)$.

$-2xz + 2y + h'(z) = 2y - 2xz$, so $h'(z) = 0$.

Since $h'(z) = 0$, $h(z)$ must be a constant function. We may take this to be $h(z) = 0$.

Thus, $f = x^2y - xz^2 + 2yz$.

The Fundamental Theorem of Line Integrals:

Suppose $\mathbf{F}$ is a conservative vector field with potential function f. Let C be a simple, piecewise-smooth curve with initial point A and final point B. Then $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = f(B) - f(A)$.

- In two dimensions, if $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then $f(B)$ means $f(b_1, b_2)$, and $f(A)$ means $f(a_1, a_2)$.
- In three dimensions, if $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, then $f(B)$ means $f(b_1, b_2, b_3)$, and $f(A)$ means $f(a_1, a_2, a_3)$.

This explains why a conservative vector field is *independent of path*: The value of the integral depends only on the value of the potential function at the points A and B. The actual path from A to B is irrelevant.

In Example 1 above, we saw that the vector field $\mathbf{F}(x,y) = < 2xe^{3y} + 3x^2,\ 3x^2e^{3y} + 4y^3 >$ has potential function $f = x^2e^{3y} + x^3 + y^4$. Let us interpret this vector field as a force field. Then the work done in moving a particle through this force field from the point $A = (1,0)$ to the point $B = (2,1)$ must be $f(2,1) - f(1,0)$.

- $f(2,1) = 4e^3 + 8 + 1 = 4e^3 + 9$.
- $f(1,0) = 1 + 1 + 0 = 2$.
- Therefore, the work done is $4e^3 + 9 - 2 = 4e^3 + 7$.

In Example 2 above, we saw that the vector field $\mathbf{F}(x,y) = < x + y, x - y >$ has potential function $f = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2$. This same vector field was considered in Examples 2 and 3 of Section 16.2. There, we saw that the work done in moving a particle from the point $(1,0)$ to the point $(-1,0)$ was zero. We can now obtain that result much more efficiently, using the Fundamental Theorem of Line Integrals:

- $f(-1,0) = \frac{1}{2}$
- $f(1,0) = \frac{1}{2}$
- Therefore, the work done is $\frac{1}{2} - \frac{1}{2} = 0$.

In 16.2 Example 3, the path was a semicircle–the top half of the unit circle centered at the origin. Suppose that, instead of traveling halfway around the circle, our particle traveled only one sixth of the way around, i.e., from the point $(1,0)$ to the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then the work done would be computed as follows:

- $f(\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{1}{2}(\frac{1}{2})^2 + \frac{1}{2}\frac{\sqrt{3}}{2} - \frac{1}{2}(\frac{\sqrt{3}}{2})^2 = \frac{1}{8} + \frac{\sqrt{3}}{4} - \frac{3}{8} = \frac{\sqrt{3}-1}{4}$
- $f(1,0) = \frac{1}{2}$

- Therefore, the work done is $\frac{\sqrt{3}-1}{4} - \frac{1}{2} = \frac{\sqrt{3}-3}{4}$

In Example 3 above, we saw that the vector field $\mathbf{F}(x,y) = < 2xy - z^2,\ x^2 + 2z,\ 2y - 2xz >$ is conservative and has potential function $f = x^2y - xz^2 + 2yz$. Let us interpret this vector field as a force field. Then the work done in moving a particle through this force field from the point $A = (0,1,2)$ to the point $B = (1,2,1)$ must be $f(1,2,1) - f(0,1,2)$.

- $f(1,2,1) = 2 - 1 + 4 = 5.$
- $f(0,1,2) = 0 - 0 + 4 = 4.$
- Therefore, the work done is $5 - 4 = 1.$

On the other hand, if we have the same vector field but the points $A = (-3,-2,-1)$ and $B = (1,2,3)$, then the work will be computed as follows:

- $f(1,2,3) = 2 - 9 + 12 = 5.$
- $f(-3,-2,-1) = -18 + 3 + 4 = -11.$
- Therefore, the work done is $5 + 11 = 16.$

Line Integrals on closed curves:

In a conservative vector field, if C is a closed curve, then $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = 0$. This should be obvious, because if $A = B$, then $f(B) - f(A) = f(B) - f(B) = 0$. Thus, in a conservative force field, when a particle travels along any path and returns to its starting point, the work done is zero.

Bear in mind, if the vector field is *not* conservative, then the work done is moving a particle along any path back to its starting point may *not* be zero.

Addendum to Section 16.3

In Section 16.3, we learned the Fundamental Theorem of Line Integrals, which works for line integrals in both two dimensions and three dimensions. However, it should be noted that, in either case, the theorem deals only with a particular kind of line integral, namely, a *work* or *circulation* integral, $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

In Section 16.2, we learned of another kind of line integral, the *flux* integral, $\int_C \mathbf{F} \cdot \mathbf{n}\, ds$. (We developed this concept only in two dimensions, not in three dimensions.)

Can we develop a theory for flux integrals analogous to the theory we have developed for work or circulation integrals? Indeed we can.

In two dimensions, with $\mathbf{F} = < p,q >$, we have $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C p\, dx + q\, dy$, and $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int\limits_C p\, dy - q\, dx$. The latter may also be written as $\int\limits_C -q\, dx + p\, dy$.

Analogous to a *potential function* for $\mathbf{F}$ is a **stream function**. Whereas a potential function is a real-valued function $f(x,y)$ whose gradient is $< p,q >$, a stream function is a real-valued function $\psi(x,y)$ whose gradient is $< -q,p >$. In other words, $\nabla\psi = < -q,p >$, so $\psi_x(x,y) = -q(x,y)$ and $\psi_y(x,y) = p(x,y)$.

Whereas a vector field is said to be *conservative* if it has a potential function, a vector field is said to be **source-free** if it has a stream function.

$\mathbf{F}$ is conservative if $p_y = q_x$. $\mathbf{F}$ is source-free if $-q_y = p_x$.

Suppose $\mathbf{F}$ is a source-free vector field with stream function ψ. Let C be a simple, piecewise-smooth curve with initial point A and final point B. Then $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$.

I like to refer to $\int_C \mathbf{F} \cdot \mathbf{T}\, ds = f(B) - f(A)$ as the *First* Fundamental Theorem of Line Integrals, and to $\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \psi(B) - \psi(A)$ as the *Second* Fundamental Theorem of Line Integrals. (This is not official terminology.)

In a source-free vector field, if C is a closed curve, then $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = 0$. This should be obvious, because if $A = B$, then $\psi(B) - \psi(A) = \psi(B) - \psi(B) = 0$.