6.5 Average Value of a Function

To calculate the average value of a finite amount of numbers, $y_1, y_2, y_3, \dots\ y_n$ we use

$$y_{avg} = \frac{y_1 + y_2 + y_3 + \cdots + y_n}{n}$$

The average value of a function f defined on an interval [a, b] is more difficult because there is an infinite number of **y** values.

We define the **average value of f** on the interval [a, b] as:

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

(The proof is on page 461 in the Calc. text.)

Example: Find the average value of the function on the given interval.

$$f(t) = e^{\sin(t)} \cdot \cos(t) \quad \left[0, \frac{\pi}{2}\right]$$

$$f_{avg} = \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} e^{\sin(t)} \cdot \cos(t)\, dt = \frac{1}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} e^{\sin(t)} \cdot \cos(t)\, dt$$

Let **u** = sin(t) then **du** = cos(t)dt. When t = 0 → u = 0 and when t = $\frac{\pi}{2}$ → u = 1 so 0 and 1 are the new limits of integration.

$$f_{avg} = \frac{2}{\pi}\int_0^1 e^u du = \frac{2}{\pi}(e^u)\Big|_0^1 = \frac{2}{\pi}(e^1 - e^0) = \boldsymbol{\frac{2}{\pi}(e-1)}$$

Example: A hiking trail has an elevation given by:

$$f(x) = 60x^3 - 650x^2 + 1200x + 4500$$

Where **f** is measured in feet above sea level and **x** represents horizontal distance along the trail in miles, with $0 \le x \le 5$. What is the average elevation of the trail?

$$f_{avg} = \frac{1}{5-0}\int_0^5 (60x^3 - 650x^2 + 1200x + 4500)dx$$

$$= \frac{1}{5}\int_0^5 (60x^3 - 650x^2 + 1200x + 4500)dx$$

$$= \frac{1}{5}\left[\frac{60x^4}{4} - \frac{650x^3}{3} + \frac{1200x^2}{2} + 4500x\right]_0^5$$

$$= \frac{1}{5}\left[15(5)^4 - \frac{650}{3}(5)^3 + 600(5)^2 + 4500(5)\right]$$

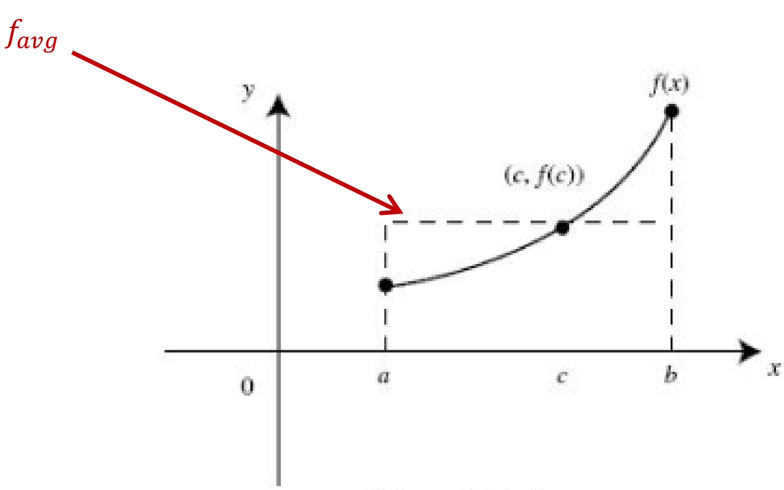
$$= \frac{1}{5}\left[15(625) - \frac{650}{3}(125) + 600(25) + 4500(5)\right]$$

$$= \frac{1}{5}\left[9375 - \frac{81250}{3} + 15000 + 22500\right]$$

$$= \frac{1}{5}\left[46875 - \frac{81250}{3}\right] = \frac{59375}{15} = \frac{\mathbf{11875}}{\mathbf{3}}\ \boldsymbol{ft.}\ = \mathbf{3958}\frac{\mathbf{1}}{\mathbf{3}}\ \boldsymbol{ft.}$$

The average elevation of the trail is slightly less than 3,960 ft.

The average value of a function bring us close to an important theoretical result. The Mean Value Theorem for Integrals says that if ***f*** is continuous on [a, b], then there is at least one point, ***c***, in the interval (a, b) such that ***f(c)*** equals the average value of ***f*** on (a, b). In other words, the horizontal line $\boldsymbol{y = f_{avg}}$ intersects the graph of ***f*** for some point ***c*** in (a, b).



Note: If ***f*** is not continuous, such a point might not exist.

Mean Value Theorem for Integrals:

Let ***f*** be continuous on the integral [a, b]. There exists a point ***c*** in [a, b] such that

$$\boldsymbol{f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx \quad \textit{that is,} \quad \int_a^b f(x)dx = f(c)(b-a)}$$

Example: Find the point(s) on the interval (0, 1) at which $f(x) = 2x(1-x)$ equals its average value on [0, 1].

First find the average value:

$$f_{avg} = \frac{1}{1-0}\int_0^1 2x(1-x)dx = \int_0^1 (2x - 2x^2)dx$$

$$= \frac{2x^2}{2} - x\Big|_0^1 = x^2 - \frac{2x^3}{3}\Big|_0^1 = 1 - \frac{2}{3} = \mathbf{\frac{1}{3}} \ (\leftarrow avg\ value)$$

Now find the point(s), **x** , where **f(x) =** $\mathbf{\frac{1}{3}}$

$$2x(1-x) = \frac{1}{3}$$

$$2x - 2x^2 = \frac{1}{3} \quad (solve\ the\ quadratic\ equation)$$

$$0 = 2x^2 - 2x + \frac{1}{3} \quad (use\ the\ quadratic\ formula)$$

$$x = \frac{2 \pm \sqrt{4 - 4(2)\left(\frac{1}{3}\right)}}{2(2)} \approx \mathbf{0.211\ and\ 0.789}$$

x = 0.211 and **0.789** is where **f(x) = 2x(1-x)** equals its average value on [0, 1].