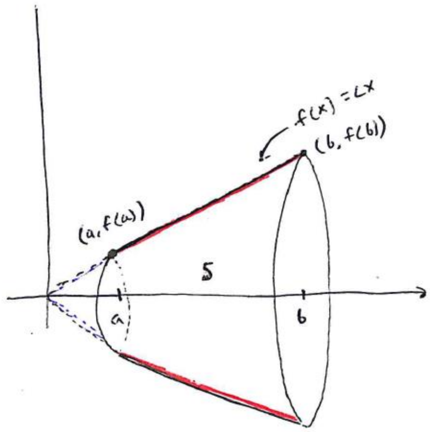
## 8.2 Area of a Surface of Revolution

In previous sections we computed the volumes of solids. In this section we will compute the area of the surface of a solid of revolution. The surface area problem is between a volume problem and the arc lenth calculation. We will use both of these ideas when finding the surface area.

To start let's consider the function $\mathbf{f(x) = c \cdot x}$ on the interval [a, b], where $0 < a < b$ and $c > 0$. When this line segment is revolved about the x-axis, it generates a cone with the top sliced off (in other words – a frustum).




Notice that the surface area **S** is the difference between $\mathbf{S_b}$ which extends over [0, b] and $\mathbf{S_a}$ which extends over [0, a]. In other words,

$$\mathbf{S = S_b - S_a}$$

From geometry we know that the surface area of a right circular cone of radius **r** and height **h** (excluding the circular base of the cone) is $\boldsymbol{\pi r\sqrt{r^2 + h^2}}$.

Notice that the radius of the cone on [0. B] is $\mathbf{r = f(b) = cb}$, and its height is **b**. This gives us:

$$S_b = \pi r\sqrt{r^2 + h^2} \;= \pi(cb)\sqrt{(cb)^2 + b^2} = \;\pi b^2 c\sqrt{c^2 + 1}$$

We get similar results for $S_a$.

$$S_a = \pi r\sqrt{r^2 + h^2} \;= \;\pi(ac)\sqrt{(ac)^2 + a^2} = \;\pi a^2 c\sqrt{c^2 + 1}$$

Therefore:

$$\begin{aligned} S &= S_b - S_a \\ &= \;\pi b^2 c\sqrt{c^2 + 1} - \;\pi a^2 c\sqrt{c^2 + 1} \\ &= \boldsymbol{\pi c\sqrt{c^2 + 1}(b^2 + a^2)} \end{aligned}$$

In addition notice that the line segment from (a, f(a)) to (b, f(b)) has length of:

$$l = \sqrt{(b - a)^2 + (bc - ac)^2} = \boldsymbol{(b - a)\sqrt{c^2 + 1}}$$

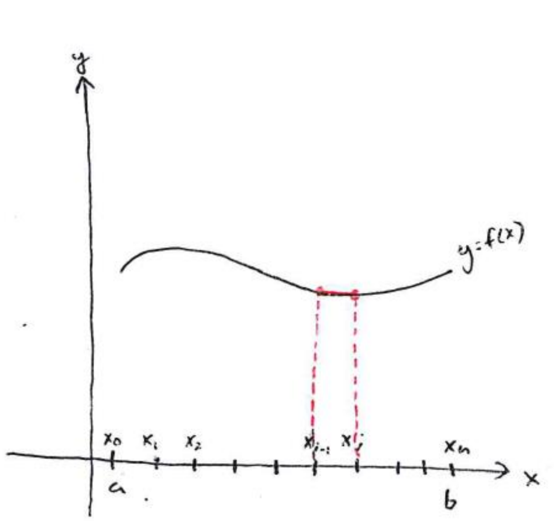
Using this we can rewrite the formula for **S.**

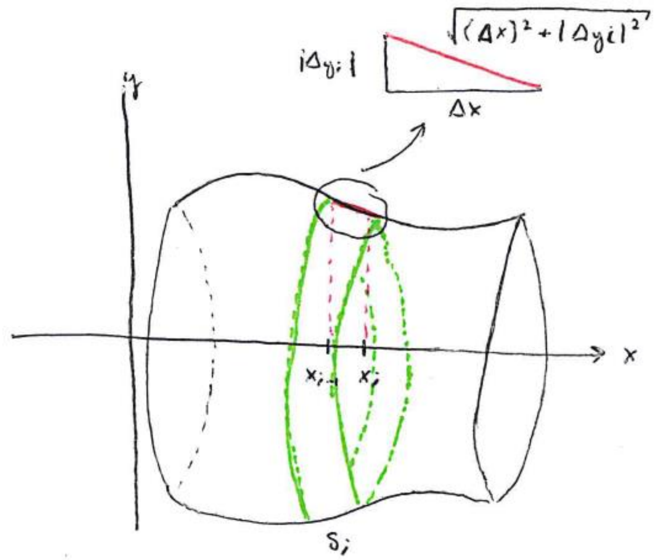
$$\begin{aligned} S &= \pi c\sqrt{c^2 + 1}(b^2 - a^2) \\ &= \pi c\sqrt{c^2 + 1}(b - a)(b + a) \\ &= \pi(bc + ac)(b - a)\sqrt{c^2 + 1} \\ &= \pi[f(b) + f(a)]l \end{aligned}$$

The **Surface Area** of the **Frustum** generated by revolving the line segment between two points, (a, f(a)) and (b, f(b)) about the x-axis is given by:

$$S = \pi l[f(b) + f(a)]$$

Using the formula above, we can now derive the general area for a surface of revolution. Let's rotate the curve $y = f(x), a \leq x \leq b$ about the x – axis, where $f$ is positive and has a continuous derivative. We subdivide the interval [a, b] into **n** subintervals of equal length: $\Delta x = \frac{b-a}{n}$. Let the endpoints be $x_0 = a, x_1, x_2, \ldots x_n = b$. The $i^{th}$ subinterval $[x_{i-1}, x_i]$ has a line segment between the two points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$. Note that the change in $y_i$, $\Delta y_i = f(x_i) - f(x_{i-1})$




The surface area $S_i = \pi(f(x_i) - f(x_{i-1}))\sqrt{(\Delta x)^2 + (\Delta y_i)^2}$

Using the ideas from previous sections, the area **S** of the entire surface of revolution is approximately the sum of each $S_i$ where $i = 1, 2, \ldots n$

$$S = \sum_{i=1}^{n} S_i$$

After using the Mean Value Theorem and as n → ∞ and Δx → 0, we obtain the following:

**Area of a Surface of Revolution:**

Let $f$ be a nonnegative function with a continuous first derivative on the interval [a, b]. The area of the surface generated whe the graph of $f$ on the interval [a, b] is revolved about the x-axis is:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(f(x)\right)^2}\,dx$$

**Example:** The graph of $f(x) = 2\sqrt{x}$ on the interval [1, 3] is revolved about the x-axis. What is the area of the surface generated?

$$f'(x) = \frac{1}{\sqrt{x}}$$

The surface area formula is:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + \left(f(x)\right)^2}\,dx$$

$$S = \int_1^3 2\pi \cdot 2\sqrt{x}\sqrt{1+\left(\frac{1}{\sqrt{x}}\right)^2}\,dx \;=\; 4\pi\int_1^3 \sqrt{x}\cdot\sqrt{1+\frac{1}{x}}\,dx \;=\; 4\pi\int_1^3 \sqrt{x}\cdot\sqrt{\frac{x+1}{x}}\,dx$$

$$= 4\pi\int_1^3 \sqrt{\frac{x^2+x}{x}}\,dx \;=\; 4\pi\int_1^3 \sqrt{x+1}\,dx$$

Use **u – substitution**: let **u** = x+1 then **du =** dx. When x = 1 → u = 2, and when x = 3 → u = 4

$$= 4\pi\int_2^4 \sqrt{u}\,du \;= 4\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^4 \;=\; \frac{8\pi}{3}\left[4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] \;=\; \frac{8\pi}{3}\left[8-\sqrt{8}\right] \approx \mathbf{43.33\overline{3}}$$

With Leibniz notation, the formula becomes:

$$\boldsymbol{S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\;dx}$$

If the curve **x = g(y)** on the interval [c, d] is revolved about the **y – axis**, the area of the surface is

$$\boldsymbol{S = \int_c^d 2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}dy}$$

**Example:** Find the area of the surface generated when the given curve is revolved about the **y – axis.**

$$y = (3x)^{\frac{1}{3}}\;\; on\;\; \left[0,\frac{8}{3}\right]$$

Since the curve is being revolved about the y – axis we need to rewrite the curve in terms of **x.** $x = \frac{y^3}{3}$

When x = 0 → $y = 0$ and when x = $\frac{8}{3}$ → y = 2. $\frac{dx}{dy} = y^2$. Using the surface area formula we have:

$$S = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\;dy$$

$$S = \int_0^2 2\pi\left(\frac{y^3}{3}\right)\sqrt{1+(y^2)^2}\,dy \;=\; \frac{2}{3}\pi\int_0^2 y^3\sqrt{1+y^4}\,dy$$

Using **u – sub.:** let **u** = $1 + y^4$ ⇒ **du** = $4y^3$ ⇒ $\frac{1}{4}$**du** = $y^3$ when y = 0 → $u = 1$ and when y = 2 → u = 17

$$S = \frac{2}{3}\pi\int_1^{17}\sqrt{u}\frac{1}{4}du \;=\; \frac{2}{3}\cdot\frac{1}{4}\int_1^{17} u^{\frac{1}{2}}du \;=\; \frac{1}{6}\pi\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{17} \;=\; \frac{\pi}{9}\left[17^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \boldsymbol{\frac{\pi}{9}\left[\sqrt{4913}-1\right]}$$

From the last section we were given that the arc length is:

$$L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

… which is part of the formula for the area of a surface:

$$S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

We rewrite this as:

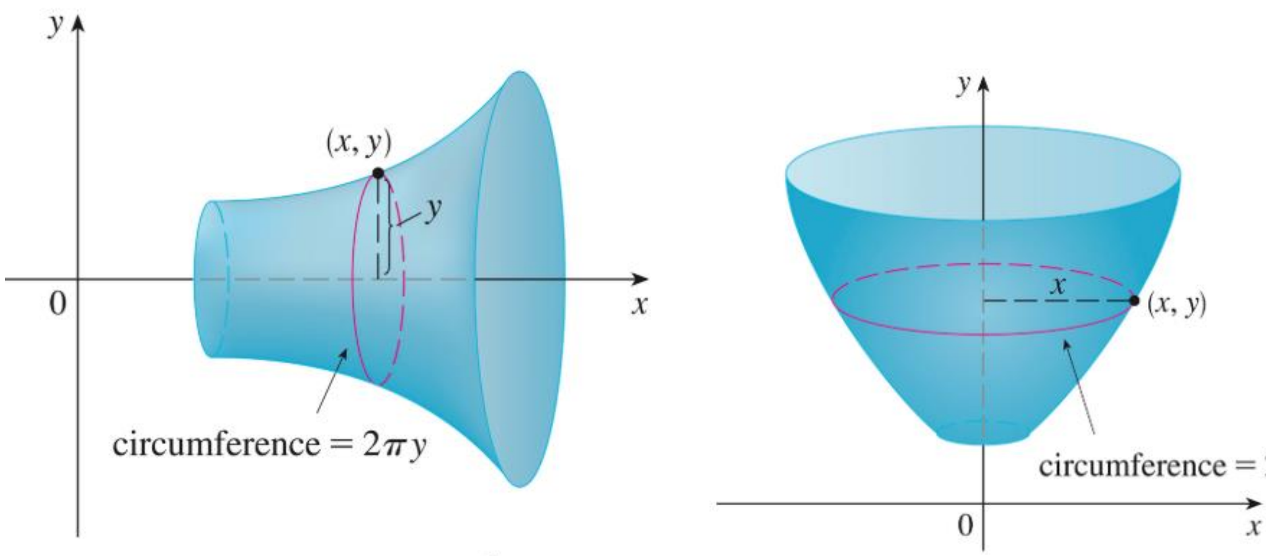
$$S = \int_a^b 2\pi f(x)ds \quad where \;\; ds = \sqrt{1 + [f'(x)]^2}\, dx$$

Similarly for the rotation about the **y – axis**:

$$S = \int_a^b 2\pi g(y)ds \;\; where \;\; ds = \sqrt{1 + [g'(y)]^2}$$

There formulas can be remembered by think of $\boldsymbol{2\pi f(x)}$ or $\boldsymbol{2\pi g(y)}$ as the circumference of a circle traced out of the point (x, y). Notice that **f(x)** and **g(y)** determine the radii.

Consider the figures below:




(a) Rotation about $x$-axis: $S = \int 2\pi y\, ds$ (b) Rotation about $y$-axis: $S = \int 2\pi x\, ds$

**Example:** The given curve is rotated about the y – axis. Find the area of the surface. $x = \sqrt{a^2 - y^2}$, $0 \le y \le \frac{a}{2}$ $\quad \frac{dx}{dy} = \frac{1}{2}(a^2 - y^2)^{-\frac{1}{2}} \cdot (-2y) = \frac{-y}{\sqrt{a^2 - y^2}}$ (No problem for the specified domain.)

$$S = \int_0^{\frac{a}{2}} 2\pi \sqrt{a^2 - y^2} \cdot \sqrt{1 + \left[\frac{-y}{\sqrt{a^2 - y^2}}\right]^2}\, dy \;=\; 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{1 + \frac{y^2}{a^2 - y^2}}\, dy$$

$$= 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{\frac{a^2 - y^2 + y^2}{a^2 - y^2}}\, dy = 2\pi \int_0^{\frac{a}{2}} \sqrt{a^2 - y^2} \cdot \sqrt{\frac{a^2}{a^2 - y^2}}\, dy$$

$$= 2\pi \int_0^{\frac{a}{2}} a\, dy \;=\; 2\pi [ay]_0^{\frac{a}{2}} \;= 2\pi \left[a \cdot \frac{a}{2} - 0\right] \;=\; \boldsymbol{a^2 \pi}$$