3.3 Derivatives of Trigonometric Functions

In this section we will learn some of the derivative of the trigonometric functions: sine, cosine, tangent, cosecant, secant, and cotangent.

A review of the trigonometric functions is given in Appendix D.

Note: When using trigonometric functions we are always using **radian** angle measures unless otherwise noted.

Let's start this section with the derivative of the function ***f(x) = sin(x).*** Use the definition of the derivative to find ***f '(x)*** if ***f(x) = sin(x)***.

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} \right]$$

$$f'(x) = \lim_{h \to 0} \left[\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \right]$$

$$f'(x) = \lim_{h \to 0} \sin(x) \cdot \lim_{h \to 0} \frac{\cos(h)-1}{h} + \lim_{h \to 0} \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$\sin(x) \quad \cdot \quad 0 \quad + \quad \cos(x) \quad \cdot \quad 1$$

$$f'(x) = \sin(x) \cdot 0 + \cos(x) \cdot 1$$

$$\boldsymbol{f'(x) = \cos(x)}$$

The Derivative of the Sine Function:

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

Using the definition of the derivative and a similar method, we can prove the derivative of ***f(x) = cos(x).***

The Derivative of the Cosine Function:

$$\frac{d}{dx}[\cos(x)] = \sin(x)$$

Example: If $f(x) = \frac{\sin(x)}{\cos(x)}$, *find* $f'(x)$ Using the quotient rule ***let g(x) = sin(x)*** and ***h(x) = cos(x)***

g'(x) = cos(x) ***h'(x) = - sin(x)***

Then $f'(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot \sin(x)}{cos^2 x}$

$= \frac{cos^2 x + sin^2 x}{cos^2 x}$ (remember $cos^2 x + sin^2 x = 1$)

$= \frac{1}{cos^2 x}$

$\boldsymbol{f'(x) = sec^2 x}$ Since $f(x) = \frac{\sin(x)}{\cos(x)} = \tan(x)$, this brings us to another derivative rule:

The derivative of the Tangent Function: $\frac{d}{dx}[\tan(x)] = \sec^2 x$

Example: find $f'(x)$, if

(a) $f(x) = \frac{1}{\cos(x)} = \sec(x)$ (b) $f(x) = \frac{1}{\sin(x)} = \csc(x)$ (c) $f(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$

(a) $f(x) = \frac{1}{\cos(x)}$ (Use quotient rule)

$$= \frac{\cos(x) \cdot 0 - 1(-\sin(x))}{\cos^2 x}$$

$$= \frac{\sin(x)}{\cos^2 x} = \frac{\sin(x)}{\cos(x) \cdot \cos(x)} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} = \tan(x)\sec(x)$$

The Derivative of the Secant Function: $\frac{d}{dx}[\sec(x)] = \tan(x)\sec(x)$

(b) $f'(x) = \frac{\sin(x)(0) - 1(\cos(x))}{\sin^2 x}$

$$= \frac{-\cos(x)}{\sin(x) \cdot \sin(x)} = \frac{-\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)} = -\cot(x)\csc(x)$$

The Derivative of the Cosecant Function: $\frac{d}{dx}[\csc(x)] = -\cot(x)\csc(x)$

(c) $f'(x) = \frac{\sin(x) \cdot (-\sin(x)) - (\cos(x))(\cos(x))}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$

$\boldsymbol{f'(x) = -\csc^2 x}$

The Derivative of the Cotangent Function: $\frac{d}{dx}[\cot(x)] = -\csc^2 x$

Example: Find the 17th derivative of ***cos(x)***. Let's see if we can find a pattern:

f(x) = cos(x)
f'(x) = - sin(x)
f''(x) = - cos(x)
f'''(x) = sin(x)
$f^{(4)}$(x) = cos(x)

> We see that there is a cycle occurring. Notice that $f^{(n)}$ = cos(x) whenever **n** is a multiple of 4. If you divide the derivative by 4, the remainder is the **n**th derivative. 17÷4 = 4 with a remainder of 1. Therefore the 17th derivative of cos(x) equals the 1st derivative of cos(x). Similar results happen with sin(x).

Example: Find the following limits.

(a) $\lim_{x\to 0} \frac{\sin(5x)}{3x}$ and (b) $\lim_{x\to 0} \frac{\sin(x)}{x+\tan(x)}$

(a) Multiply the numerator and denominator by 5 – in other words, multiply by a form of one $\left(\frac{5}{5}\right)$.

$\lim_{x\to 0} \frac{\sin(5x)}{3x} \cdot \frac{5}{5} = \lim_{x\to 0} \frac{5 \cdot \sin(x)}{5 \cdot 3x} = \lim_{x\to 0} \frac{5}{3} \cdot \frac{\sin(5x)}{5x} = \frac{5}{3} \cdot \lim_{x\to 0} \frac{\sin(5x)}{5x}$ (remember $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$) so

$$\lim_{x\to 0} \frac{\sin(5x)}{3x} = \frac{5}{3}$$

(b) Divide the numerator and denominator by ***x***. $\lim_{x\to 0} \frac{\sin(x)}{x+\tan(x)} = \lim_{x\to 0} \frac{\frac{\sin(x)}{x}}{\frac{x+\tan(x)}{x}} = \frac{\lim_{x\to 0} \frac{\sin(x)}{x}}{\lim_{x\to 0} 1 + \frac{\tan(x)}{x}}$

$$= \frac{\lim_{x\to 0}\frac{\sin(x)}{x}}{\lim_{x\to 0} 1+\frac{\frac{\sin(x)}{\cos(x)}}{x}} = \frac{\lim_{x\to 0}\frac{\sin(x)}{x}}{\lim_{x\to 0}\ 1+\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}} = \frac{\lim_{x\to 0}\frac{\sin(x)}{x}}{\lim_{x\to 0}\ 1+\lim_{x\to 0}\frac{\sin(x)}{x}\cdot \lim_{x\to 0}\cos(x)} = \frac{1}{1+1\cdot 1} = \mathbf{\frac{1}{2}}$$

$$\mathbf{\lim_{x\to 0}\frac{\sin(x)}{x+\tan(x)} = \frac{1}{2}}$$