12.5) Equations Of Planes and Lines

In two-dimensional space, an equation of the form $Ax + By = C$, where A and B are *not both* zero, gives us a **line**. In three-dimensional space, an equation of the form $Ax + By + Cz = D$, where A, B, and C are *not all* zero, gives us a **plane**, rather than a line. (However, both equations are referred to as *linear equations*.)

Usually, the equations $Ax + By = C$ and $Ax + By + Cz = D$ are written so that the first nonzero coefficient is positive rather than negative. There is no mathematical reason for this; it's just a nicety.

1. **Equations of Planes**:

$x = 0$ is the equation of the y,z plane. Technically, this is $1x + 0y + 0z = 0$.

$y = 0$ is the equation of the x,z plane. Technically, this is $0x + 1y + 0z = 0$.

$z = 0$ is the equation of the x,y plane. Technically, this is $0x + 0y + 1z = 0$.

The x,y plane, the x,z plane, and the y,z planes are referred to as the **coordinate planes** because each is determined by two of our *coordinate axes* (the x,y plane is determined by the x and y axes, the x,z plane is determined by the x and z axes, and the y,z plane is determined by the y and z axes).

For any nonzero real number k, $x = k$ is a plane *parallel* to the plane $x = 0$, $y = k$ is a plane *parallel* to the plane $y = 0$, and $z = k$ is a plane *parallel* to the plane $z = 0$.

The x,y plane or any plane parallel to it ($z = 0$ or $z = k$) is a said to be a **horizontal plane**. Every horizontal plane *perpendicular* to the z axis and *parallel* to the x and y axes. Furthermore, it is *perpendicular* to the x,z plane and to the y,z plane.

Any plane *perpendicular* to the x,y plane is said to be a **vertical** plane. Every vertical plane is *parallel* to the z axis.

- The y,z plane or any plane parallel to it ($x = 0$ or $x = k$) is a vertical plane. It is *perpendicular* to the x axis and is *parallel* to the y axis as well as the z axis. Furthermore, it is *perpendicular* to the x,z plane as well as the x,y plane.
- The x,z plane or any plane parallel to it ($y = 0$ or $y = k$) is a vertical plane. It is *perpendicular* to the y axis and is *parallel* to the x axis as well as the z axis. Furthermore, it is *perpendicular* to the y,z plane as well as the x,y plane.
- A plane such as $2x + 3y = 6$ is an **oblique vertical plane**–i.e., a plane that is vertical (and hence parallel to the z axis), but neither parallel nor perpendicular to either the x axis or the y axis. It is not perpendicular to either the x,z plane or the y,z plane. We also refer to this as an **oblique plane parallel to the z axis**.

- A plane such as $2x + 3y = 0$ is an oblique vertical plane *passing through the origin*, $(0,0,0)$.

Notice that *every* vertical plane has an equation of the form $Ax + By = D$, where A and B are not both zero.

The *orthogonal projection* of a vertical plane onto the x,y plane is a line in the x,y plane, whose equation is of the form $Ax + By = D$, where A and B are not both zero.

- The projection of the y,z plane, $x = 0$, onto the the x,y plane is the *line* $x = 0$, which coincides with the y axis and is considered a “vertical” line in the x,y plane.
- For nonzero k, the projection of the plane $x = k$ onto the the x,y plane is the *line* $x = k$, which is parallel to the y axis and is considered a “vertical” line in the x,y plane.
- The projection of the x,z plane, $y = 0$, onto the the x,y plane is the *line* $y = 0$, which coincides with the x axis and is considered a “horizontal” line in the x,y plane.
- For nonzero k, the projection of the plane $y = k$ onto the the x,y plane is the *line* $y = k$, which is parallel to the x axis and is considered a “horizontal” line in the x,y plane.
- The projection of the oblique vertical plane $2x + 3y = 6$ onto the x,y plane is the *line* $2x + 3y = 6$, which is not parallel to either the x or y axis and is considered an “oblique” line in the x,y plane.
- The projection of the oblique vertical plane $2x + 3y = 0$ onto the x,y plane is the *line* $2x + 3y = 0$, which is an oblique line passing through the origin, $(0,0)$.

Conversely, the vertical plane $Ax + By = D$ can be thought of as the *orthogonal projection* of the line $Ax + By = D$ in the x,y plane *into* x,y,z space, parallel to the z axis.

The *orthogonal projection* of a horizontal plane onto the x,z plane is a *horizontal line* in the x,z plane.

- The projection of the x,y plane, $z = 0$, onto the the x,z plane is the *line* $z = 0$, which coincides with the z axis in the x,z plane.
- For nonzero k, the projection of the horizontal plane $z = k$ onto the the x,z plane is the *line* $z = k$, which is parallel to the z axis in the x,z plane.

The *orthogonal projection* of a horizontal plane onto the y,z plane is a *horizontal line* in the y,z plane.

- The projection of the x,y plane, $z = 0$, onto the the y,z plane is the *line* $z = 0$, which coincides with the z axis in the x,z plane.
- For nonzero k, the projection of the horizontal plane $z = k$ onto the the y,z plane is the *line* $z = k$, which is parallel to the z axis in the x,z plane.

The *orthogonal projection* of the y,z plane, or any plane parallel to it, onto the x,z plane is a *vertical line* in the x,z plane.

- The projection of the y,z plane, $x = 0$, onto the the x,z plane is the *line* $x = 0$, which coincides with the z axis in the x,z plane.

- For nonzero $k,$ the projection of the vertical plane $x = k$ onto the the x,z plane is the *line* $x = k,$ which is parallel to the z axis in the x,z plane.

The *orthogonal projection* of the x,z plane, or any plane parallel to it, onto the y,z plane is a *vertical line* in the y,z plane.

- The projection of the x,z plane, $y = 0,$ onto the the y,z plane is the *line* $y = 0,$ which coincides with the z axis in the y,z plane.
- For nonzero $k,$ the projection of the vertical plane $y = k$ onto the the y,z plane is the *line* $y = k,$ which is parallel to the z axis in the y,z plane.

An equation such as $3x + 5z = 15$ represents an oblique line in the x,z plane, and it represents a plane in three-dimensional space, which is the orthogonal projection of that line perpendicular to the x,z plane, or parallel to the y axis. (Conversely, the line is the projection of that plane onto the x,z plane.) We refer to this type of plane as an **oblique plane parallel to the y axis**.

An equation such as $2y + 7z = 14$ represents an oblique line in the y,z plane, and it represents a plane in three-dimensional space, which is the orthogonal projection of that line perpendicular to the y,z plane, or parallel to the x axis. (Conversely, the line is the projection of that plane onto the y,z plane.) We refer to this type of plane as an **oblique plane parallel to the x axis**.

An equation such as $2x + 3y + 4z = 12$ represents a **strictly oblique plane**. It is not parallel to *any* of the three axes.

$2x + 3y + 4z = 0$ is a strictly oblique plane *passing through the origin*.

2. **Using Vectors to Write Equations of Planes**:

We learned in Section 12.4 that a plane is uniquely determined by two nonzero, non-parallel vectors and a given point. In other words, given nonzero, non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ and a point $P,$ there is a unique plane to which $\mathbf{a}$ and $\mathbf{b}$ both belong and which contains $P.$ If $\overrightarrow{PA}$ represents $\mathbf{a}$ and $\overrightarrow{PB}$ represents $\mathbf{b},$ we name the plane $PAB.$ $\mathbf{a} \times \mathbf{b}$ is nonzero and is orthogonal to $PAB.$

Recall that when we are dealing with nonzero vectors, the words *orthogonal* and *perpendicular* are synonymous. Another synonym for these words is *normal*, i.e., two nonzero vectors are **normal** if and only if they are perpendicular or orthogonal. Thus, we can say that $\mathbf{a} \times \mathbf{b}$ is *normal to* $PAB,$ and we therefore call it a *normal vector* for that plane.

In general, a nonzero vector is said to be **normal to** a plane, and is called a **normal vector** for the plane, if and only if it is orthogonal to every vector belonging to that plane. For example, the vector $\mathbf{i}$ is normal to the y,z plane, the vector $\mathbf{j}$ is normal to the x,z plane, and the vector $\mathbf{k}$ is normal to the x,y plane.

A vector is a normal vector for PAB if and only if it is parallel to $\mathbf{a} \times \mathbf{b}$ (i.e., if and only if it is a nonzero scalar multiple of $\mathbf{a} \times \mathbf{b}$).

A plane is *uniquely determined* by a point in the plane and a vector normal to the plane. In other words, given a point and a normal vector, there is one and only one plane containing that point and having that normal vector.

Let $P_0 = (x_0, y_0, z_0)$ be a given point in a plane, and let $\mathbf{n} = < a, b, c >$ be a normal vector for the plane. Let $P = (x, y, z)$ be any point in the plane. Then $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$, so $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

$\overrightarrow{P_0P} = < x - x_0, y - y_0, z - z_0 >$, so $\mathbf{n} \cdot \overrightarrow{P_0P} = a(x - x_0) + b(y - y_0) + c(z - z_0)$. Consequently:

- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ Call this **Point**, **Normal Vector Form**
- $ax + by + cz = ax_0 + by_0 + cz_0$

The latter equation corresponds to **Standard Form**, $Ax + By + Cz = D$, where A, B, C are the same as a, b, c, respectively, and $D = ax_0 + by_0 + cz_0$. (However, we may need to divide both sides of the equation by -1 to ensure that the first nonzero coefficient is positive, if we choose to follow that convention.)

Our book writes $ax + by + cz + d = 0$, so $d = -D$ and $D = -d$.

If $\mathbf{r}$ is the position vector for the point P, i.e., $\mathbf{r} = < x, y, z >$, then $Ax + By + Cz = \mathbf{n} \cdot \mathbf{r}$.

If $\mathbf{r}_0$ is the position vector for the point P_0, i.e., $\mathbf{r}_0 = < x_0, y_0, z_0 >$, then $D = \mathbf{n} \cdot \mathbf{r}_0$.

Thus, Standard Form can be condensed as $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$.

Usually, we use $\mathbf{a} \times \mathbf{b}$ as our normal vector $\mathbf{n}$.

Let us revisit Examples Four and 10 of Section 12.4. We had $\mathbf{a} = < 4, 1, 3 >$ and $\mathbf{b} = < 3, 4, 5 >$, with $\mathbf{a} \times \mathbf{b} = < -7, -11, 13 >$. Let $\mathbf{a}$ and $\mathbf{b}$ be placed at the point $P = (5, 2, 6)$, so that $A = (9, 3, 9)$ and $B = (8, 6, 11)$. We asserted that the equation of PAB is $7x + 11y - 13z = -21$. Let us now see how that equation was obtained. Using $(5, 2, 6)$ as (x_0, y_0, z_0) and $< -7, -11, 13 >$ as $< a, b, c >$, we have:

$-7(x - 5) - 11(y - 2) + 13(z - 6) = 0$ (This is Point, Normal Vector Form)

$-7x + 35 - 11y + 22 + 13z - 78 = 0$

$-7x - 11y + 13z - 21 = 0$

$-7x - 11y + 13z = 21$

$7x + 11y - 13z = -21$ (We divided by -1 to obtain a positive leading coefficient)

We could also have obtained this from the formula $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$.

$\mathbf{n} \cdot \mathbf{r} = < -7, -11, 13 > \cdot < x, y, z > = -7x - 11y + 13z$

$\mathbf{n} \cdot \mathbf{r}_0 = < -7, -11, 13 > \cdot < 5, 2, 6 > = -35 - 22 + 78 = 21$

Thus, we obtain $-7x - 11y + 13z = 21$

Dividing by -1 on both sides gives us $7x + 11y - 13z = -21$.

3. Intercepts of Planes:

If a plane intersects the x axis at a unique point, that point is known as the plane's x **intercept**. If a plane does not intersect the x axis at a unique point, then it does not have an x intercept (i.e., the x intercept is *undefined*). This could occur because the plane fails to intersect the x axis at all, or because the plane contains the x axis. (The planes $z = 5$ and $z = 0$ illustrate these two possibilities.)

The plane's y and z intercepts are defined similarly.

Every point on the x axis has 0 as its y and z coordinates. Every point on the y axis has 0 as its x and z coordinates. Every point on the z axis has 0 as its x and y coordinates. Hence, assuming each intercept exists, we may find each intercept by substituting 0 for the other two coordinates. In other words:

- We find the x intercept by substituting 0 for y and z.
- We find the y intercept by substituting 0 for x and z.
- We find the z intercept by substituting 0 for x and y.

The plane $Ax + By + Cz = D$ has an x intercept if and only if A is nonzero. It has a y intercept if and only if B is nonzero. It has a z intercept if and only if C is nonzero.

- When A is nonzero, the x intercept is $(\frac{D}{A}, 0, 0)$, casually referred to as $\frac{D}{A}$.
- When B is nonzero, the y intercept is $(0, \frac{D}{B}, 0)$, casually referred to as $\frac{D}{B}$.
- When C is nonzero, the z intercept is $(0, 0, \frac{D}{C})$, casually referred to as $\frac{D}{C}$.

For example, the plane $3x - 5z = 15$ (or $3x + 0y - 5z = 15$) has x and z intercepts but no y intercept (the plane does not intersect the y axis). Its x intercept is $(5, 0, 0)$ and its z intercept is $(0, 0, -3)$. We may casually refer to these intercepts as 5 and -3, respectively.

If $D = 0$, then any intercept that exists must be the origin, $(0, 0, 0)$. We may casually refer to each intercept as 0.

For example, the plane $3x - 5z = 0$ has $(0, 0, 0)$ as its x and z intercepts. Its y intercept is undefined (the plane contains the y axis). Casually, we may say that its x and z intercepts are 0.

If $D = 0$ and A, B, and C are all nonzero, then all three intercepts are the origin.

For example, the plane $7x - 5y + 11z = 0$ has $(0, 0, 0)$ as its x, y, and z intercepts. Casually, we may say that its x, y, and z intercepts are 0.

If A, B, C, and D are all nonzero, the plane has x, y, and z intercepts that are three distinct points. In this case, we can use the three intercepts to sketch the plane, by drawing a triangular "wedge" of the plane.

For example, the plane $2x + 3y + 4z = 12$ has x intercept $(6, 0, 0)$, y intercept $(0, 4, 0)$, and z intercept $(0, 0, 3)$. (We may casually refer to these intercepts as 6, 4, and 3, respectively.) We may plot these three points and then connect them with line segments to draw a triangular wedge of the plane.

If a plane has nonzero x, y, z intercepts p, q, r, respectively, then its equation can be written as $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$. (The derivation of this formula is as follows: $p = \frac{D}{A}$, so $A = \frac{D}{p}$. $q = \frac{D}{B}$, so $B = \frac{D}{q}$. $r = \frac{D}{C}$, so $C = \frac{D}{r}$. By substitution, the equation $Ax + By + Cz = D$ may be written $\frac{D}{p}x + \frac{D}{q}y + \frac{D}{r}z = D$. Dividing out the D gives us $\frac{1}{p}x + \frac{1}{q}y + \frac{1}{r}z = 1$.)

For example, the plane $2x + 3y + 4z = 12$ could be written as $\frac{x}{6} + \frac{y}{4} + \frac{z}{3} = 1$.

4. Parametric, Symmetric, and Vector Equations of Lines:

In Section 12.2, we discussed how to write **parametric equations** for a line in either two-dimensional space or three-dimensional space. These equations express the line in terms of an **independent parameter**, usually t. Let us review this subject...

Given a line in the x, y plane, we choose any two distinct points on the line, $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$. Let $a = x_1 - x_0$, and let $b = y_1 - y_0$. Then the line has *parametric equations* $x = x_0 + at$, $y = y_0 + bt$, where $t \in (-\infty, \infty)$. Note that a and b cannot *both* be zero. The point (x_0, y_0) corresponds to $t = 0$, and the point (x_1, y_1) corresponds to $t = 1$.

Given a line in x, y, z space, we choose any two distinct points on the line, $P_0 = (x_0, y_0, z_0)$ and $P_1 = (x_1, y_1, z_1)$. Let $a = x_1 - x_0$, let $b = y_1 - y_0$, and let $c = z_1 - z_0$. Then the line has *parametric equations* $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, where $t \in (-\infty, \infty)$. Note that a, b, and c cannot *all* be zero. The point (x_0, y_0, z_0) corresponds to $t = 0$, and the point (x_1, y_1, z_1) corresponds to $t = 1$.

When we have written parametric equations for a line, we may say that we have *parameterized* the line, or have *represented the line parametrically*, or have adopted a *parameterization* for the line. This all depends on the choice of the points P_0 and P_1. Since P_0 and P_1 are chosen arbitrarily, there are infinitely many different ways to parameterize a line.

If we restrict the parameter so $t \in [0, 1]$, we obtain the line segment $\overline{P_0 P_1}$.

$x_0 + at$ is a function of t, giving us the x coordinate of the point on the line generated by a given value of t. We may refer to this function as $x(t)$. Thus, we have $x(t) = x_0 + at$. Similarly, we may write $y(t) = y_0 + bt$ and, in the three-dimensional case, $z(t) = z_0 + ct$.

Note that $x(0) = x_0$, $x(1) = x_1$, $y(0) = y_0$, $y(1) = y_1$, $z(0) = z_0$, and $z(1) = z_1$.

Any real value of t generates a unique point on the line, denoted P_t. In the two-dimensional case, $P_t = (x(t), y(t))$. In the three-dimensional case, $P_t = (x(t), y(t), z(t))$.

The point P_0 may be referred to as the **initial point** of the line, and the point P_1 may be referred to as the **unitary point** of the line. (These designations apply only in the context of the chosen parameterization.)

(When dealing with an oriented line segment, we refer to its second endpoint as the terminal point. When dealing with a line, however, it would not make sense to refer to any point as a terminal point, because the line does not "stop" at any point. Hence, you should *not* refer to P_1 as the terminal point. Instead, we name it the unitary point, because "unitary" derives from the word "unit," which means 1, which is the value of t that generates this point.)

A given parameterization implies a particular *orientation* of the line. The line's *forward* or *positive direction* is the direction we follow when moving from P_0 to P_1. Hence, a line with a parameterization may be referred to as an *oriented line* or a *directed line*.

For a parameterized two-dimensional line, if a and b are both nonzero, then the parametric equations $x = x_0 + at,\ y = y_0 + bt$ can be solved for t, giving us the equations $t = \frac{x - x_0}{a}$ and $t = \frac{y - y_0}{b}$. It follows that $\frac{x - x_0}{a} = \frac{y - y_0}{b}$. This is known as the **symmetric equation** of the line.

For a parameterized three-dimensional line, if a, b, and c are all nonzero, then the parametric equations $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$ can be solved for t, giving us the equations $t = \frac{x - x_0}{a}$, $t = \frac{y - y_0}{b}$, and $t = \frac{z - z_0}{c}$. It follows that:

- $\frac{x - x_0}{a} = \frac{y - y_0}{b}$
- $\frac{x - x_0}{a} = \frac{z - z_0}{c}$
- $\frac{y - y_0}{b} = \frac{z - z_0}{c}$

These three equations are known as the **symmetric equations** of the line. We summarize them by writing $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$.

Let $\mathbf{r}_0$ be the position vector for the point P_0, and let $\mathbf{r}_1$ be the position vector for the point P_1.

- In two-dimensional space, $\mathbf{r}_0 = < x_0, y_0 >$ and $\mathbf{r}_1 = < x_1, y_1 >$. So $\mathbf{r}_1 - \mathbf{r}_0 = < x_1 - x_0, y_1 - y_0 > = < a, b >$.
- In three-dimensional space, $\mathbf{r}_0 = < x_0, y_0, z_0 >$ and $\mathbf{r}_1 = < x_1, y_1, z_1 >$. So $\mathbf{r}_1 - \mathbf{r}_0 = < x_1 - x_0, y_1 - y_0, z_1 - z_0 > = < a, b, c >$.

$\mathbf{r}_1 - \mathbf{r}_0$ is a nonzero vector and is referred to as a **direction vector** for the line. We shall denote it as $\mathbf{v}$. When placed at any point on the line, $\mathbf{v}$ points in the line's forward or positive direction. In particular, when placed at tail P_0, its tip is P_1, so $\mathbf{v}$ is represented by the directed line segment $\overrightarrow{P_0P_1}$.

The components of $\mathbf{v}$ are known as the **direction numbers** of the line. In other words, in the two-dimensional case, the direction numbers are a and b, while in the three-dimensional case, the direction numbers are a, b, and c.

Let $\mathbf{r}(t)$ be the position vector for the point P_t.

- In two-dimensional space, $\mathbf{r}(t) = < x(t), y(t) >$.
- In three-dimensional space, $\mathbf{r}(t) = < x(t), y(t), z(t) >$.

In two-dimensional space, $\mathbf{r}(t) = < x(t), y(t) > = < x_0 + at, y_0 + bt > =$
$< x_0, y_0 > + \ t < a, b > = \mathbf{r}_0 + t\mathbf{v}$.

In three-dimensional space, $\mathbf{r}(t) = < x(t), y(t), z(t) > = < x_0 + at, y_0 + bt, z_0 + ct > =$
$< x_0, y_0, z_0 > + \ t < a, b, c > = \mathbf{r}_0 + t\mathbf{v}$.

In both two and three dimensions, we have $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} =$
$\mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) =$
$\mathbf{r}_0 + t\mathbf{r}_1 - t\mathbf{r}_0 =$
$\mathbf{r}_0 - t\mathbf{r}_0 + t\mathbf{r}_1 =$
$(1 - t)\mathbf{r}_0 + t\mathbf{r}_1$

In summary, a parameterized line may be represented by the following **vector equations**:

- $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ 2D or 3D
- $\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$ 2D or 3D
- $\mathbf{r}(t) = < x_0 + at, y_0 + bt >$ 2D
- $\mathbf{r}(t) = < x_0 + at, y_0 + bt, z_0 + ct >$ 3D
- $\mathbf{r}(t) = < x_0, y_0 > + \ t < a, b >$ 2D
- $\mathbf{r}(t) = < x_0, y_0, z_0 > + \ t < a, b, c >$ 3D
- $\mathbf{r}(t) = (x_0 + at)\mathbf{i} + (y_0 + bt)\mathbf{j}$ 2D
- $\mathbf{r}(t) = (x_0 + at)\mathbf{i} + (y_0 + bt)\mathbf{j} + (z_0 + ct)\mathbf{k}$ 3D

5. Intersections of Lines:

When we are dealing with two lines, each represented parametrically, it is best to use a different independent parameter for each line. Usually, we use the parameter s for one line and the parameter t for the other line. We may represent line L_1 by $\mathbf{r}_1(s)$ and line L_2 by $\mathbf{r}_2(t)$.

In the x,y plane, $\mathbf{r}_1(s) = < x_1(s), y_1(s) > = < x_{0_1} + a_1 s, y_{0_1} + b_1 s >$, and $\mathbf{r}_2(t) = < x_2(t), y_2(t) > = < x_{0_2} + a_2 t, y_{0_2} + b_2 t >$. Line L_1 has direction vector $\mathbf{v}_1 = < a_1, b_1 >$, while line L_2 has direction vector $\mathbf{v}_2 = < a_2, b_2 >$.

In x,y,z space, $\mathbf{r}_1(s) = < x_1(s), y_1(s), z_1(s) > = < x_{0_1} + a_1 s, y_{0_1} + b_1 s, z_{0_1} + c_1 s >$, and $\mathbf{r}_2(t) = < x_2(t), y_2(t), z_2(t) > = < x_{0_2} + a_2 t, y_{0_2} + b_2 t, z_{0_2} + c_2 t >$. Line L_1 has direction vector

$\mathbf{v}_1 = < a_1, b_1, c_1 >$, while line L_2 has direction vector $\mathbf{v}_2 = < a_2, b_2, c_2 >$.

$\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel vectors if and only if L_1 and L_2 are either the same line or parallel lines.

To find the intersection of the two lines, we solve a system of linear equations.

In the x,y plane, we solve the following system (two equations in two unknowns):

- $x_1(s) = x_2(t)$, or $x_{0_1} + a_1 s = x_{0_2} + a_2 t$, or $a_1 s - a_2 t = x_{0_2} - x_{0_1}$
- $y_1(s) = y_2(t)$, or $y_{0_1} + b_1 s = y_{0_2} + b_2 t$, or $b_1 s - b_2 t = y_{0_2} - y_{0_1}$

In x,y,z space, we solve the following system (three equations in two unknowns):

- $x_1(s) = x_2(t)$, or $x_{0_1} + a_1 s = x_{0_2} + a_2 t$, or $a_1 s - a_2 t = x_{0_2} - x_{0_1}$
- $y_1(s) = y_2(t)$, or $y_{0_1} + b_1 s = y_{0_2} + b_2 t$, or $b_1 s - b_2 t = y_{0_2} - y_{0_1}$
- $z_1(s) = z_2(t)$, or $z_{0_1} + c_1 s = z_{0_2} + c_2 t$, or $c_1 s - c_2 t = z_{0_2} - z_{0_1}$

If the system has infinitely many solutions, then L_1 and L_2 are the *same line*. If the system has a unique solution, then L_1 and L_2 cross at a *unique point*. If the system has no solution, then L_1 and L_2 are *disjoint* (i.e., non-intersecting).

In the x,y plane, two lines are disjoint if and only if they are *parallel*. In x,y,z space, however, disjoint lines are either *parallel* or *skew*. (Two lines are said to be **skew** with respect to each other if they are non-intersecting and non-parallel; skew lines can arise only in three-dimensional space, never in two-dimensional space.) Once you have determined that a pair of three-dimensional lines is disjoint (because the three-by-two system of equations has no solution), you can then decide whether the lines are parallel or skew by comparing their direction vectors (if the vectors are parallel, then so are the lines, but if the vectors are non-parallel, then the lines are skew).

To solve the three-by-two system of equations, we pick any two of the three equations, giving us a two-by-two system. We solve this two-by-two system, using the algebra techniques of substitution or elimination by addition. If the two-by-two system has no solution, then neither does the three-by-two system. On the other hand, if the two-by-two system has a solution, then we must check that this ordered pair also satisfies the third equation. If it does, then we have found the solution of the three-by-two system, but if it does not, then the three-by-two system has no solution.

Example (a): Say we have the lines $\mathbf{r}_1(s) = < 6 + 3s, 4 - s, 10 + 4s >$ and $\mathbf{r}_2(t) = < 1 + t, -5 + 5t, 2 + 2t >$. We must solve the linear system $6 + 3s = 1 + t$, $4 - s = -5 + 5t$, and $10 + 4s = 2 + 2t$. We may rewrite these equations as $3s - t = -5$, $s + 5t = 9$, and $4s - 2t = -8$. Let us choose the first two equations. If we elminate s, we obtain $t = 2$. Plugging in 2 for t, we obtain $s = -1$. Now we check that these values satisfy the third equation (which they do). Now we may find the point of intersection. We may either plug $s = -1$ into the first vector equation, or we may plug $t = 2$ into the second vector equation. Either way, we get the point $(3, 5, 6)$. This is the intersection point for the two lines.

Example (b): Say we have the lines $\mathbf{r}_1(s) = < -2s, 5+3s, 4-2s >$ and $\mathbf{r}_2(t) = < 3+4t, 1-6t, 4t >$. We must solve the linear system $-2s = 3+4t$, $5+3s = 1-6t$, $4-2s = 4t$. We may rewrite these equations as $2s+4t = -5$, $3s+6t = -4$, and $2s+4t = 4$. If we use elimination on the first and third equations, we obtain the result $0 = 9$, indicating the system has no solution (the lines do not intersect). Are the lines parallel or skew? The direction vector for the first line is $\mathbf{v}_1 = < -2, 3, -2 >$. The direction vector for the second line is $\mathbf{v}_2 = < 4, -6, 4 >$. Note that $\mathbf{v}_2 = -2\mathbf{v}_1$, or $\mathbf{v}_1 = -\frac{1}{2}\mathbf{v}_2$. Since the direction vectors are parallel, the lines are parallel.

Example (c): Say we have the lines $\mathbf{r}_1(s) = < -6+2s, 3+s, 1+2s >$ and $\mathbf{r}_2(t) = < 4-t, 5+t, -2+3t >$. We must solve the linear system $-6+2s = 4-t$, $3+s = 5+t$, $1+2s = -2+3t$. We may rewrite these equations as $2s+t = 10$, $s-t = 2$, and $2s-3t = -3$. Let us choose the first two equations. If we elminate t, we obtain $s = 4$. Plugging in 4 for s, we obtain $t = 2$. Now we check that these values satisfy the third equation. They do not! The system has no solution, and the lines do not intersect. Are the lines parallel or skew? The direction vector for the first line is $\mathbf{v}_1 = < 2, 1, 2 >$. The direction vector for the second line is $\mathbf{v}_2 = < -1, 1, 3 >$. These vectors are not scalar multiples of each other, so the lines are skew.

6. **Miscellaneous Other Topics**:

The distance between the plane $Ax+By+Cz = D$, or $ax+by+cz+d = 0$, and a point not in the plane, $P_1 = (x_1, y_1, z_1)$, is $\frac{|Ax_1+By_1+Cz_1-D|}{\sqrt{A^2+B^2+C^2}}$, or $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$. Here is a way to remember this. Let $\mathbf{n} = < a, b, c >$ be the normal vector for the plane, and let $\mathbf{r}_1$ be the position vector for P_1, i.e., $\mathbf{r}_1 = < x_1, y_1, z_1 >$. Then the distance is $\frac{|\mathbf{n} \cdot \mathbf{r}_1 + d|}{n}$.

Let L be a given line containing the point $P_0 = (x_0, y_0, z_0)$, and let $P_1 = (x_1, y_1, z_1)$ be a point not lying on L. Let $\mathbf{u} = \overrightarrow{P_0P_1} = < x_1-x_0, y_1-y_0, z_1-z_0 >$, and let $\mathbf{v}$ be a direction vector for L. Then the distance between P_1 and L is $\frac{|\mathbf{u} \times \mathbf{v}|}{v}$.

Two planes with parallel normal vectors are either the same plane or parallel planes. To see which, write the equations so their left sides are identical. The planes $Ax+By+Cz = D_1$ and $Ax+By+Cz = D_2$ are parallel if and only if $D_1 \neq D_2$. The distance between these two planes is $\frac{|D_1-D_2|}{\sqrt{A^2+B^2+C^2}}$. This can also be expressed as $\frac{|D_1-D_2|}{n}$, where $\mathbf{n} = < A, B, C >$ is the normal vector for the two planes.

- The planes $6x-9y-12z = 15$ and $8x-12y-16z = 20$ have normal vectors $< 6, -9, -12 >$ and $< 8, -12, -16 >$, respectively. These vectors are parallel, because each is a scalar multiple of the other (the latter is $\frac{4}{3}$ times the former, and the former is $\frac{3}{4}$ times the latter). Multiplying the second equation by $\frac{3}{4}$, we obtain the equivalent equation $6x-9y-12z = 15$. Thus, the two planes are the same plane.
- The planes $4x-6y+8z = 5$ and $6x-9y+12z = 7$ have normal vectors $< 4, -6, 8 >$ and $< 6, -9, 12 >$, respectively. These vectors are parallel, because each is a scalar multiple of the other (the latter is $\frac{3}{2}$ times the former, and the former is $\frac{2}{3}$ times the

latter). Multiplying the second equation by $\frac{2}{3}$, we obtain the equivalent equation $4x - 6y + 8z = \frac{14}{3}$. Now we see that we have parallel planes, with $D_1 = 5$ and $D_2 = \frac{14}{3}$. $D_1 - D_2 = 5 - \frac{14}{3} = \frac{15}{3} - \frac{14}{3} = \frac{1}{3}$, and $n = \sqrt{A^2 + B^2 + C^2} = \sqrt{4^2 + (-6)^2 + 8^2} = \sqrt{116}$ or $2\sqrt{29}$, so the distance between the planes is $\frac{1}{3\sqrt{116}}$ or $\frac{1}{6\sqrt{29}}$ or $\frac{\sqrt{29}}{147}$.

Two skew lines can be thought of as two lines lying in parallel planes. Consequently, the distance between the two skew lines is the same as the distance between those two planes.

Given the equations of two skew lines, we can find the parallel planes containing them as follows. Let $\mathbf{v}_1$ be a direction vector for the first line, and let $\mathbf{v}_2$ be a direction vector for the second line. Then $\mathbf{v}_1 \times \mathbf{v}_2$ can serve as a normal vector for both planes. Using any point on the first line, we can write an equation for the first plane in Point, Normal Vector Form, and then we do likewise for the second plane.

The **angle between two non-parallel planes** is the angle between their normal vectors. Thus, if the planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, then the angle is $\cos^{-1} \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{n_1 n_2}$.

Two planes are perpendicular if and only if their normal vectors are orthogonal. In other words, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

The intersection of two non-parallel planes is a line. If the planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, then $\mathbf{n}_1 \times \mathbf{n}_2$ can serve as a direction vector for the line (the line is in both planes, so it is perpendicular to both normal vectors). To write an equation for the line, we also need a point lying on the line. The simplest way of finding a point on the line is to take the equations of the two planes, substitute 0 for one of the three variables, and then solve the resulting two-by-two system of linear equations.