16.1) Vector Fields

We have seen vector-valued functions with *two*-dimensional or *three*-dimensional domains–namely, we have studied **gradient functions**.

- Given a real-valued function with a two-dimensional domain, $z = f(x,y)$, its gradient function is $\nabla f(x,y) = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > = < f_x(x,y), f_y(x,y) >$. This is a function which has a *two-dimensional domain* and whose value is a *two-dimensional vector*.
- Given a real-valued function with a three-dimensional domain, $w = f(x,y,z)$, its gradient function is $\nabla f(x,y,z) = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > = < f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) >$. This is a function which has a *three-dimensional domain* and whose value is a *three-dimensional vector*.

Recall that at any given point in its domain, if ∇f is nonzero, then ∇f points in the direction in which f is *increasing* most rapidly. The magnitude of ∇f is the *maximum rate of increase* of the function at the given point.

A gradient function is an example of a vector field. A **vector field** is a vector-valued function where the domain and the vector value have the *same dimension*–either both two-dimensional or both three-dimensional. (The dimension could be higher than three, but we will not address that situation.)

An example of a three-dimensional vector field would be $\mathbf{F}(x,y,z) = < 2x, 3y, -5z >$. It applies a three-dimensional vector to every point in three-dimensional space. For example, at the point $(-1,4,-3)$, it applies the vector $< -2, 12, 15 >$.

At any given point in its domain, a vector field produces a vector which, if nonzero, can be depicted as a directed line segment. We place this directed line segment so that its *tail* is the point in question.

- If a two-dimensional vector field produces a nonzero vector $< a,b >$ at a point (x_1,y_1), then we depict this vector as having tail (x_1,y_1) and head $(x_1 + a, y_1 + b)$.
- If a three-dimensional vector field produces a nonzero vector $< a,b,c >$ at a point (x_1,y_1,z_1), then we depict this vector as having tail (x_1,y_1,z_1) and head $(x_1 + a, y_1 + b, z_1 + c)$.

In the preceding example, $\mathbf{F}(-1,4,-3) = < -2, 12, 15 >$. We would depict this vector as having tail $(-1,4,-3)$ and head $(-3,16,12)$.

Since a gradient function is a vector field, it may be referred to as a **gradient vector field**, or simply as a **gradient field**. Thus, ∇f is said to be the *gradient field* for f. Conversely, f is said to be a *potential function* for ∇f.

In general, for any given vector field, a **potential function** is a real-valued function whose gradient is the given vector field. The potential function and the vector field have the same domain.

A vector field may or may not have a potential function. If it does, then it is called a **conservative vector field**.

Obviously, a gradient vector field is automatically conservative, but there are other kinds of vector fields that may or may not be conservative.

Another important type of vector field is a **force field**. This may be either two-dimensional or three-dimensional. The force is typically either gravitational or electro-magnetic.

- A **two-dimensional force field** is a vector field that associates a two-dimensional *force vector* with each point in the x,y plane: $\mathbf{F}(x,y) = < p(x,y), q(x,y) >$ $= p(x,y)\mathbf{i} + q(x,y)\mathbf{j}$.
- A **three-dimensional force field** is a vector field that associates a three-dimensional *force vector* with each point in x,y,z space: $\mathbf{F}(x,y,z) =$ $< p(x,y,z), q(x,y,z), r(x,y,z) > = p(x,y,z)\mathbf{i} + q(x,y,z)\mathbf{j} + r(x,y,z)\mathbf{k}$.

When analyzing a force field, if the force vector at any given point is nonzero, then its directed line segment points in the direction in which the force is being exerted at that point. The length of the directed segment is the magnitude (or strength) of the force.

The simplest force field would apply a uniform force at every point–i.e., no matter where we are in the field, the field produces a force of fixed magnitude with a fixed direction. This is rarely the case. With most force fields, as we move from one point to another, the force varies in its magnitude or its direction (or both).

Example: Consider the two-dimensional force field $\mathbf{F}(x,y) = < 6x - 4y, 9x - 6y >$. The force applied at the point $(4,3)$ is $< 12, 18 >$ or $12\mathbf{i} + 18\mathbf{j}$, the force applied at the point $(-2,-5)$ is $< 8, 12 >$ or $8\mathbf{i} + 12\mathbf{j}$, and the force applied at the point $(2,3)$ is $< 0, 0 >$ or $0\mathbf{i} + 0\mathbf{j}$. Of these three forces, the first two can be depicted as directed line segments. The first would have tail $(4,3)$ and head $(16,21)$. The second would have tail $(-2,-5)$ and head $(6,7)$.

According to Newton's theory of gravitation, the planet Earth exerts a *gravitational force field,* which can be represented as a three-dimensional vector field. We set up our x,y,z coordinate system so that the origin is the center of the Earth. Let M be the mass of the Earth, and let m be the mass of a particle located on or near the surface of the Earth. Let G be the gravitational constant (as determined in Newton's Law of Universal Gravitation). If the particle is located at position (x,y,z), then the force of Earth's gravity acting on the particle is $\mathbf{F}(x,y,z) = -mMG(x^2 + y^2 + z^2)^{-3/2} < x,y,z >$.

Suppose we have a river or stream in which water is flowing at different velocities at different points. We can then associate a particular velocity vector with each point, giving us a vector field. This kind of vector field is known as a **velocity field**. It is a two-dimensional field if we study only the surface of the water, and it is a three-dimensional field if we study the water beneath the surface. Alternatively, we could be studying *wind velocity* in the air; this would necessarily be a three-dimensional vector field. In general, velocity fields arise in connection with **fluid flow**, and the fluid may be either a liquid, such as water, or a gas, such as air (although, technically, air is a combination of different gases).

If a vector field $\mathbf{F}$ is conservative, then a potential function for $\mathbf{F}$ may be denoted f.

- If $\mathbf{F}(x,y)$ is conservative, then $f(x,y)$ is a function such that $\nabla f(x,y) = \mathbf{F}(x,y)$.
- If $\mathbf{F}(x,y,z)$ is conservative, then $f(x,y,z)$ is a function such that $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$.

The following two statements are equivalent:

- $f(x,y)$ is a potential function for $\mathbf{F}(x,y)$.
- $\mathbf{F}(x,y)$ is the gradient field for $f(x,y)$.

The following two statements are equivalent:

- $f(x,y,z)$ is a potential function for $\mathbf{F}(x,y,z)$.
- $\mathbf{F}(x,y,z)$ is a gradient field for $f(x,y,z)$.

The gravitational force field $\mathbf{F}(x,y,z) = -mMG(x^2+y^2+z^2)^{-3/2} < x,y,z >$ is conservative. It has potential function $f(x,y,z) = mMG(x^2+y^2+z^2)^{-1/2} = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$.

The vector field $\mathbf{F}(x,y,z) = < yz, xz, xy >$ has potential function $f(x,y,z) = xyz$, because $\nabla f(x,y,z) = < yz, xz, xy >$.

Potential functions are not unique. In fact, a conservative vector field has infinitely many potential functions.

- If $f(x,y)$ is a potential function for $\mathbf{F}(x,y)$, then so is $f(x,y) + C$, where C is any real number.
- If $f(x,y,z)$ is a potential function for $\mathbf{F}(x,y,z)$, then so is $f(x,y,z) + C$, where C is any real number.

Thus, potential functions are very much like antiderivatives (or indefinite integrals).

The vector field $\mathbf{F}(x,y,z) = < yz, xz, xy >$ has general potential function $f(x,y,z) = xyz + C$.

The vector field $\mathbf{F}(x,y) = < 10x, 6y^2 >$ has general potential function $f(x,y) = 5x^2 + 2y^3 + C$.

There are two essential questions we will not address until later:

1. Given a vector field, how do we decide whether or not it is conservative–i.e., whether or not it *has* a potential function?
2. If a vector field *is* conservative, how do we *find* a potential function for it?

For now, you are not being asked to find a potential function for a vector field, you are merely being asked to *confirm* that a specified function is indeed a potential function for a given vector field.

Example: Confirm that the vector field $\mathbf{F}(x,y,z) = < 2xy^3 + 3x^2z^5, 3x^2y^2 + 2yz^4, 4y^2z^3 + 5x^3z^4 >$ has potential function $f(x,y,z) = x^2y^3 + y^2z^4 + x^3z^5$.