4.2 The Mean Value Theorem (MVT)

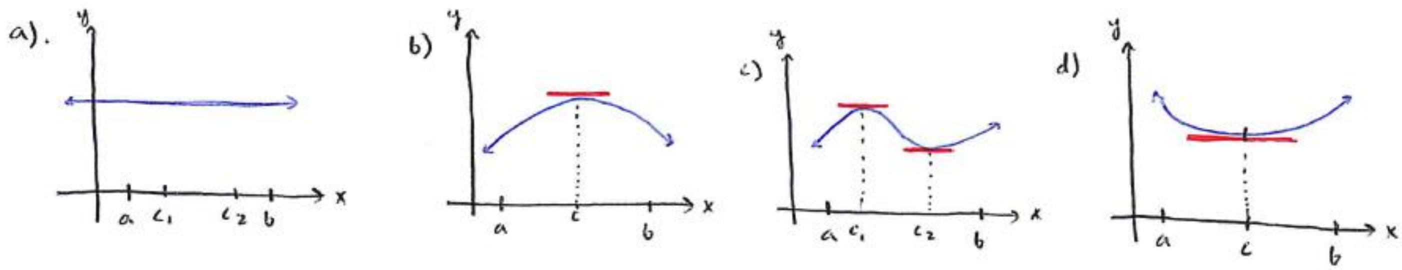
Before we introduce The Mean Value Theorem, we will introduce the theorem that The Mean Value Theorem was developed from, this is Rolle's Theorem.

Rolle's Theorem: Let ***f*** be a function that satisfies the following three hypotheses:

1. ***f*** is continuous on the closed interval [a, b]
2. ***f*** is differentiable on the open interval (a, b)
3. ***f(a) = f(b)***

Then there is a number ***c*** in (a, b) such that ***f'(c) = 0***.

Rolle's Theorem give us the following scenarios:



Example: Using Rolle's Theorem, prove that $f(x) = 2x^2 - 4x + 5,\ [-1, 3]$ has a number ***c*** that satisfies ***f'(c) = 0***.

To do this we must show that the function ***f*** satisfies the three hypotheses.

1. ***f(x)*** is continuous because all polynomial functions are continuous.
2. It is differential on the interval (-1, 3).
3. $f(-1) = 2(-1)^2 - 4(-1) + 5 = 11$ and $f(3) = 2(3)^2 - 4(3) + 5 = 11$

Therefore there is a number ***c***, in (-1, 3) such that ***f'(c) = 0***.

Find ***c***.

$f'(x) = 4x - 4 = 0 \quad \Rightarrow \quad 4x = 4 \quad \Rightarrow \quad x = 1$ There is a horizontal tangent where ***x = 1***.

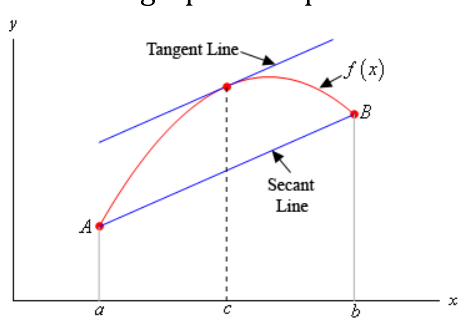
The main use of Rolle's Theorem is to prove the Mean Value Theorem (MVT)

The Mean Value Theorem (MVT): Let ***f*** be a function that satisfies the following hypotheses:

1. ***f*** is continuous on the closed interval [a, b].
2. ***f*** is differentiable on the open interval (a, b).

Then there is a number ***c*** in (a, b) such that: $f'(c) = \frac{f(b)-f(a)}{b-a}$ or $f(b) - f(a) = f'(c)(b - a)$

Below is a graphical representation of the Mean Value Theorem.



Notice what the MVT says: There is some ***c*** in (a, b) that has the slope of its tangent line equal to the slope of the secant line from point ***a*** to ***b***.

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Example: Assume that the function satisfies the two requirements for the Mean Value Theorem. Find the number $\boldsymbol{c}$ that satisfies the conclusion of the Mean Value Theorem if $\boldsymbol{f(x) = \sqrt{x}}$ on **[0, 4]**

Find the slope of the secant line. $m = \frac{f(b)-f(a)}{b-a} = \frac{2-0}{4-0} = \frac{1}{2}$

Then set the derivative = slope of the secant and solve for $\boldsymbol{x}$.

$f'(x) = \frac{1}{2}$

$\frac{1}{2\sqrt{x}} = \frac{1}{2} \Rightarrow 2\sqrt{x} = 2 \Rightarrow \sqrt{x} = 1 \Rightarrow x = \pm 1$ (but -1 is not in [0, 4] so ...) $\boldsymbol{x = 1}$

Translation: The "c" value at which the function $\boldsymbol{f(x) = \sqrt{x}}$ has a tangent whose slope is $\frac{1}{2}$ in the closed interval [0, 4] is 1.

Example: If ***f(1) = 10*** and ***f ′(x) ≥ 2*** for **1 ≤ x ≤ 4**, how **small** can **f(4)** possibly be?

(Using equation 2 of the Mean Value Theorem we have...)

$$f(4) - f(1) = f'(c)(4-1)$$
$$f(4) - f(1) = 4f'(c) - f'(c)$$
$$f(4) = 4f'(c) - f'(c) + f(1)$$

(We're given that $f(1) = 10$ and $f'(c) \geq 2$):

$$f(4) = 4f'(c) - f'(c) + f(1)$$

(The smallest value for $f'(c)$ is 2 ...) $f(4) \geq 4(2) - 2 + 10$

$$\boldsymbol{f(4) \geq 16}$$

Therefore $\boldsymbol{f(4)}$ can be as small as 16.

Theorem: If $f'(x) = 0$ for all $\boldsymbol{x}$ in an interval (a, b), then $\boldsymbol{f}$ is constant on (a, b). Meaning $\boldsymbol{f}$ is a constant function.