Section 2.6 Limits at Infinity: Horizontal Asymptotes

In this section we observe how horizontal asymptotes arise when we speak in terms of limits. We will also discuss the precise definition of a horizontal asymptote and examine different examples.

In past sections, we let x approach a specific number and the value of y became large (positive or negative). In this section we observe how letting the x value become large (positive or negative) affects the value of y.

Let's start by looking at an example.

Example: Analyze the behavior of $\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{x^3+4}}{\boldsymbol{x^3-2}}$ as x approaches large (positive or negative) values.
From Algebra and Precalculus we know that ***f(x)*** has a horizontal asymptote at y = 1. Let's make a table with large (positive and negative) values.

x	f(x)	x	f(x)
1	-5	-1	-1
5	1.0488	-5	0.9528
10	1.0060	-10	0.9940
20	1.0008	-20	0.9993
50	1.0000	-50	1.0000
100	1.0000	-100	1.0000
1000	1.0000	-1000	1.0000

As ***x*** gets larger (positive and negative) the values of ***f(x)*** gets closer and closer to 1. Mathematically, for ***x*** approaches positive large number we write this as $\mathbf{lim}_{x\to\infty} \boldsymbol{f}(\boldsymbol{x}) = \mathbf{1}$ or $\mathbf{lim}_{x\to\infty} \frac{\boldsymbol{x^3+4}}{\boldsymbol{x^3-2}} = \mathbf{1}$

Intuitive Definition of a **Limit at Infinity:** Let f be a function defined on some interval (a, ∞). Then

$$\mathbf{lim}_{x\to\infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}$$

means that the values of ***f(x)*** can be made arbitrarily close to L by replacing x to be sufficiently large.

Similarly, in the last example we observe that as **x** approaches negative large numbers, ***f(x)*** gets close to1.

Mathematically, $\mathbf{lim}_{x\to-\infty} \boldsymbol{f}(\boldsymbol{x}) = \mathbf{1}$ or $\mathbf{lim}_{x\to-\infty} \frac{\boldsymbol{x^3+4}}{\boldsymbol{x^3-2}} = \mathbf{1}$

Definition: Let ***f*** be a function defined on some interval $(-\infty, a)$. Then $\mathbf{lim}_{x\to-\infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}$ means that the values of ***f(x)*** can be made arbitrarily close to L by requiring x to be sufficiently large negative.

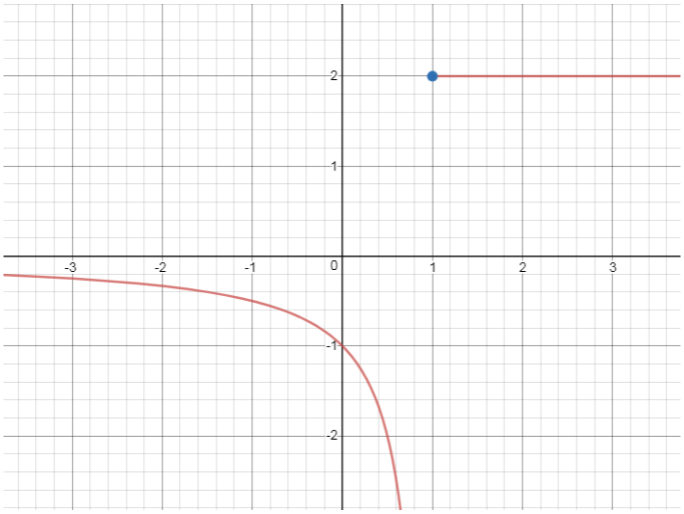
By combining the last two definitions we get the following:

Definition: The line ***y = L*** is called a horizontal asymptote of the curve ***y = f(x)*** if either

$$\mathbf{lim}_{x\to\infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L} \quad \text{or} \quad \mathbf{lim}_{x\to-\infty} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{L}$$

Example: Find $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, *and* $\lim_{x\to 1} f(x)$ where $f(x) = \begin{Bmatrix} \frac{1}{x-1} & if\ x < 1 \\ 2 & if\ x \geq 1 \end{Bmatrix}$

Plot the function.



$$\lim_{x\to\infty} f(x) = 2$$

$$\lim_{x\to-\infty} f(x) = 0$$

$$\lim_{x\to 1} f(x) = DNE$$

Theorem: If $r > 0$ is a rational number, then $\mathbf{\lim_{x\to\infty} \frac{1}{x^r} = 0}$

If $r > 0$ is a rational number such that x^r is defined for all x, then $\mathbf{\lim_{x\to-\infty} \frac{1}{x^r} = 0.}$

Example: Evaluate $\lim_{x\to-\infty} \frac{x-2}{x^2+1}$ From Precalculus we know that the answer is 0, but let's do some algebra to show why.

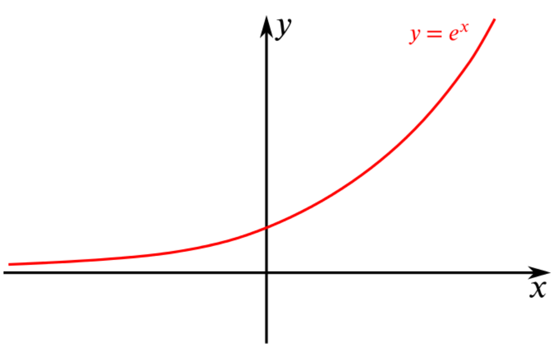
$$\lim_{x\to\infty} \frac{x-2}{x^2+1} = \lim_{x\to-\infty} \frac{\frac{x-2}{x^2}}{\frac{x^2+1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x}-\frac{2}{x^2}}{1+\frac{1}{x^2}} = \frac{\lim_{x\to-\infty} \frac{1}{x}-\frac{2}{x^2}}{\lim_{x\to-\infty} 1+\frac{1}{x^2}} = \frac{\lim_{x\to-\infty}\left(\frac{1}{x}\right)-\lim_{x\to-\infty}\left(\frac{2}{x^2}\right)}{\lim_{x\to-\infty}(1)+\lim_{x\to-\infty}\left(\frac{1}{x^2}\right)} = \frac{0-0}{1+0} = 0$$

Thus $\lim_{x\to-\infty} \frac{x-2}{x^2+1} = 0.$

Example: Evaluate $\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$ Divide both numerator and denominator by x^3.

$$\lim_{x\to\infty} \frac{\frac{\sqrt{1+4x^6}}{x^3}}{\frac{2-x^3}{x^3}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1+4x^6}{x^6}}}{\frac{2-x^3}{x^3}} = \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \frac{\sqrt{\lim_{x\to\infty}\frac{1}{x^6}+\lim_{x\to\infty} 4}}{\lim_{x\to\infty}\frac{2}{x^3}-\lim_{x\to\infty} 1} = \frac{\sqrt{0+4}}{0-1} = \frac{\sqrt{4}}{-1} = -2$$

Now let's analyze the natural exponential function $\boldsymbol{y = e^x}$. Its graph is plotted below.

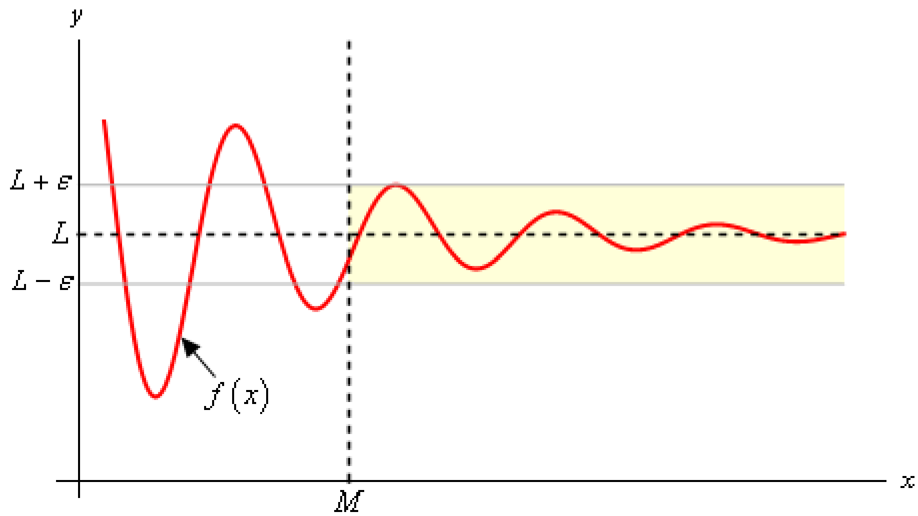


Notice that as $\boldsymbol{x}$ approaches $-\infty$, y approaches 0.

Thus $\lim_{x\to-\infty} e^x = 0$

Precise Definition of a Limit at Infinity: Let f be a function defined on some interval (a, ∞), then $\lim_{x\to\infty} f(x) = L$ means that for every $\varepsilon > 0$ there is a corresponding number M, such that If $x > M$, *then* $|f(x) - L| < \varepsilon$

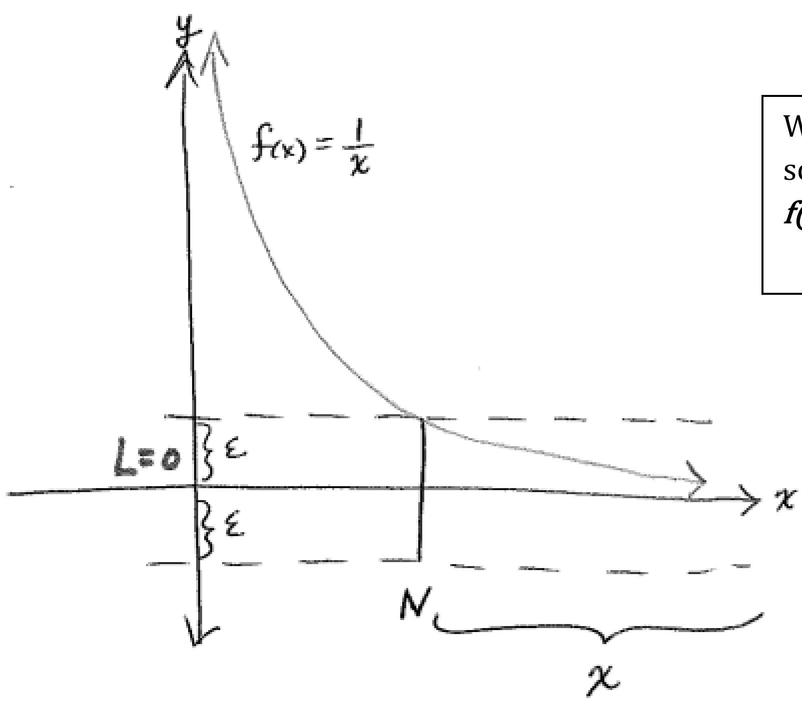
Graphically



Similarly for negative infinity:

Definition: Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = L$ means that for every $\varepsilon > 0$, there is a corresponding number N, such that If $x < N$, *then* $|f(x) - L| < \varepsilon$.

Example: Plot the idea of a limit as x approaches ∞ for the function $f(x) = \frac{1}{x}$ using the definition.



We need to find a value of x greater than some N such that the difference between $f(x)$ and the limit value, 0, is less than ε.

Notice we can find a value of N, depending on the value of ε that we might want.

Lastly, some functions do not approach a specific ***y value*** as x approaches positive or negative infinity.. Here is the definition for this situation for x approaching positive infinity.

Definition of an Infinite Limit at Infinity: Let $\boldsymbol{f}$ be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = \infty$ means that for every positive number $\boldsymbol{M}$ there is a corresponding positive number $\boldsymbol{N}$ such that **if** $\boldsymbol{x > N\ then\ f(x) > M.}$

There is a similar definition for x approaches negative infinity.