4.7 Optimization Problems

In this section we will learn how to maximize or minimize a situation. For example, we might want to maximize our profit and minimize our cost. Once we create the model for the problem we would like to analyze, we can then find the optimal solution.

Steps in Solving Optimization Problems

1. Understand the problem
   a. Identify the variables.
   b. What are the given conditions?
2. Draw a diagram.
3. Introduce notation.
   a. Assign a symbol to the quantity that needs to be optimized (maximized or minimized)
   b. Assign other variables to any unknown quantities.
4. Create your function or model for the variable that needs to be optimized in terms of the other variables.
5. Find relationships between the variables
   a. Find the domain of the function.
6. Find the absolute maximum or minimum value of the function.
   a. if the domain of the function is closed, then use the closed interval method.

Example: Of all rectangles of area 100 square units, which one has the minimum perimeter? Give the dimensions.

Area of a rectangle = base x height = bh
Perimeter of a rectangle = 2 x base + 2 x height = 2b + 2h

We want to minimize the perimeter P. We have two equations with two unknowns (variables). Since the area of 100 square units is a constant, lets rewrite the area formula of bh = 100 for b in terms of h.

$\mathbf{b} = \frac{100}{h}$ and substitute it into the perimeter equation.

$$P = 2b + 2h$$

$$P = 2\frac{100}{h} + 2h$$

$$P = \frac{200}{h} + 2h$$

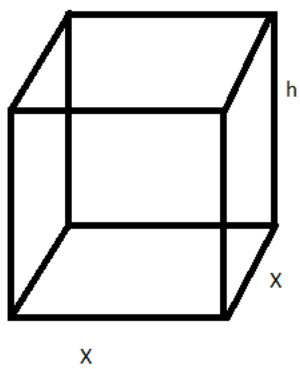
$P(h) = \frac{200+2h^2}{h}$ To find h that makes P a minimum, take the derivative of P, set it equal to 0 and solve for h.

$$P'(h) = \frac{h(4h)-(200+2h^2)(1)}{h^2} = \frac{4h^2-200-2h^2}{h^2} = \frac{2h^2-200}{h^2}$$

$$P'(h) = 0 \Rightarrow \frac{2h^2-200}{h^2} = 0 \Rightarrow h^2 = 100 \Rightarrow h = 10$$

If $h = 10$, then $b = 10$. These numbers give us a minimum perimeter.

Example: Of all boxes with a square base and a volume of 100 m^3, which one has the minimum surface area? Give the dimensions.



We have that Volume V = (x)(x)(h) (because of the square base) so
$V = x^2h = 100\ m^3$.
Surface Area, $S_A = A_{top} + A_{bottom} + A_{left} + A_{right} + A_{front} + A_{back}$
$= (x)(x) + (x)(x) + (x)(h) + (x)(h) + (x)(h) + (x)(h)$
$S_A = 2x^2 + 4xh$
We need to take the derivative of S_A but we want only one variable.
Since $x^2h = 100$, solve for h $\Rightarrow$ $h = \frac{100}{x^2}$. So substitute $h = \frac{100}{x^2}$ in for the h in $S_A = 2x^2 + 4xh$ $\quad S_A = 2x^2 + 4x\left(\frac{100}{x^2}\right) \Rightarrow \boldsymbol{S_A = 2x^2 + \frac{400}{x}}$

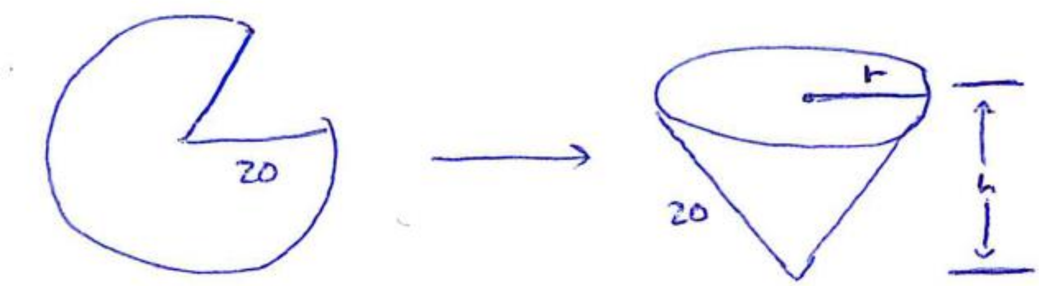
Now take the derivative of $S_A = 2x^2 + \frac{400}{x} \Rightarrow S'_A = 4x + (-1)(400)x^{-2} \Rightarrow \boldsymbol{S'_A = 4x - \frac{400}{x^2}}$

Let $S'_A = 0$ and solve for $\boldsymbol{x}$.

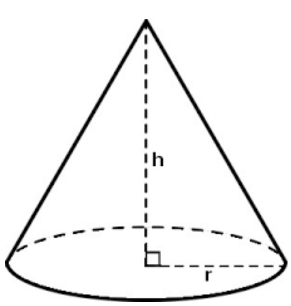
$4x - \frac{400}{x^2} = 0 \quad \Rightarrow \quad 4x = \frac{400}{x^2} \quad \Rightarrow \quad 4x^3 = 400 \Rightarrow x^3 = 100 \quad \Rightarrow \quad \boldsymbol{x = \sqrt[3]{100}}$ gives the minimum surface area.

Find $\boldsymbol{h}$ using $h = \frac{100}{x^2} = \frac{100}{100^{2/3}} = 100^{1-\frac{2}{3}} = 100^{\frac{1}{3}} = \sqrt[3]{100}$. **So the length = width = height =** $\boldsymbol{\sqrt[3]{100}}$

Example: A cone is constructed by cutting a sector from a circular sheet of metal with radius of 20. The cut sheet is then folded up and welded. Find the radius and height of the cone with maximum volume.



We have that the formula for the volume of a cone is $\boldsymbol{V = \frac{1}{3}\pi r^2 h}$. Notice that we have two variables, thus we need to make a connection between $\boldsymbol{r}$ and $\boldsymbol{h}$ so we can rewrite one variable in terms of the other.



As you can see from the picture at the left, the height, radius and the side of the cone (which is 20) form a right triangle thus we can use the Pythagorean Theorem to make a connection between $\boldsymbol{h}$ and $\boldsymbol{r}$. $\boldsymbol{h^2 + r^2 = 20^2}$ $\Rightarrow$
$h^2 + r^2 = 400 \Rightarrow r^2 = 400 - h^2 \Rightarrow r = \sqrt{400 - h^2}$. Substitute this into
$V = \frac{1}{3}\pi r^2 h$. $\quad V = \frac{1}{3}\pi\left(\sqrt{400 - h^2}\right)^2 h \Rightarrow \frac{1}{3}\pi(400 - h^2)h \Rightarrow \boldsymbol{V = \frac{\pi}{3}(400h - h^3)}$

Now find V' and set V'equal to 0 and solve for h to find the critical number(s). $\boldsymbol{V' = \frac{\pi}{3}(400 - 3h^2)}$.

$\frac{\pi}{3}(400 - 3h^2) = 0$

$(400 - 3h^2) = 0$

$400 = 3h^2$

$h^2 = \frac{400}{3}$

$\boldsymbol{h = \frac{20}{\sqrt{3}}}$ Now find r using $\mathbf{r = \sqrt{400 - h^2}}$

$r = \sqrt{400 - \left(\frac{20}{\sqrt{3}}\right)^2} \Rightarrow \sqrt{400 - \frac{400}{3}} \Rightarrow \sqrt{\frac{1200}{3} - \frac{400}{3}} \Rightarrow \sqrt{\frac{800}{3}} \Rightarrow \boldsymbol{\frac{20\sqrt{2}}{\sqrt{3}}}$

So the cone with maximum volume has a radius of $\boldsymbol{\frac{20\sqrt{2}}{\sqrt{3}}}$ and a height of $\boldsymbol{\frac{20}{\sqrt{3}}}$.