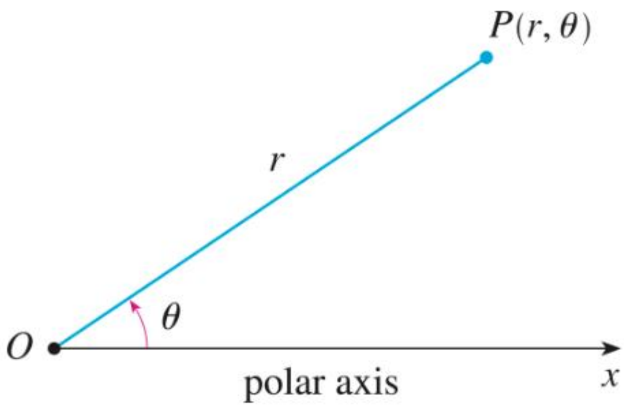
10.3 Polar Coordinates

In this section we introduce the **polar coordinate system.** The Origin in the Cartesian Coordinate System is called the **Pole** in the polar coordinate system. We will label it "O". The positive x-axis is the **polar axis.** If **P** is any other point in the plane (other than the pole), let **r** be the distance from **O** to **P** and let **Θ** be the angle in radians between the polar axis and the line **OP.**



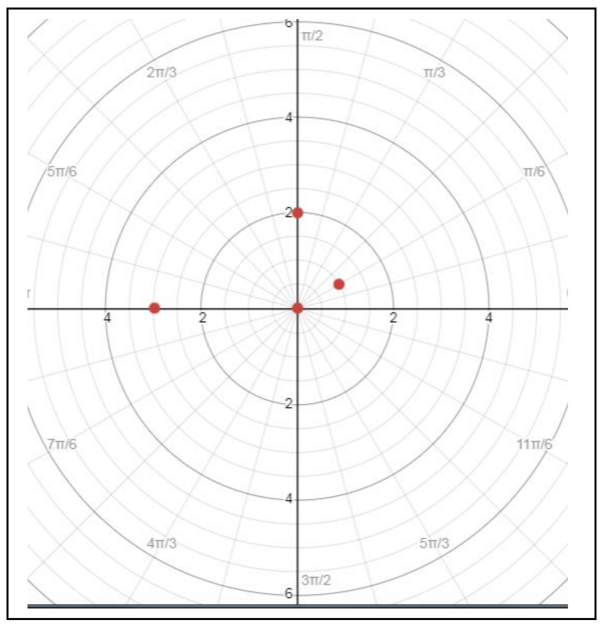
The point **P** is represented by the ordered pair **(r, Θ)**, which are the polar coordinates.

- The angle, Θ, is positive if measured counter-clockwise from the polar axis
- The angle, Θ, is negative if measured clockwise from the polar axis.

If **P = 0**, then **r = 0** and **(0, Θ)** represents any point at the pole for different values of Θ.

Example: Plot the following points.

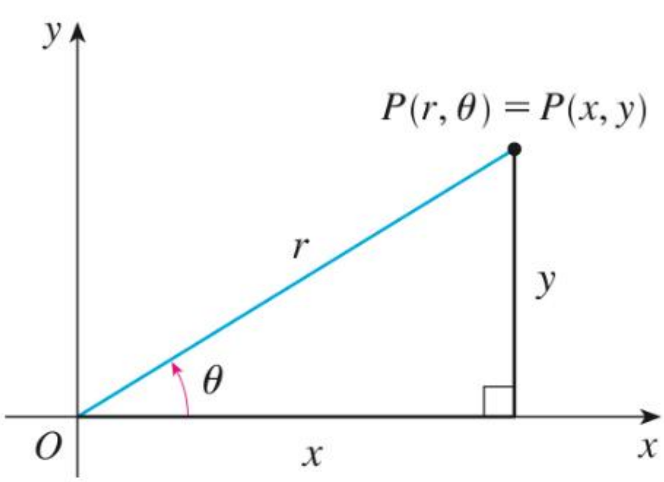
a) $\left(-2, \frac{3\pi}{2}\right)$ b) $\left(1, \frac{\pi}{6}\right)$ c) $\left(0, \frac{\pi}{2}\right)$ d) $(3, -3\pi)$



If the point **P** has Cartesian coordinates (x, y) and polar coordinates (r, Θ), then from the figure below we achieve the following:

$x = r\cos(\theta)$

$y = r\sin(\theta)$



The equations above help us to convert from polar to cartesian.

The equations below help us to convert from cartesian to polar.

$$x^2 + y^2 = r^2 \qquad \tan(\theta) = \frac{y}{x}$$

Polar Curves

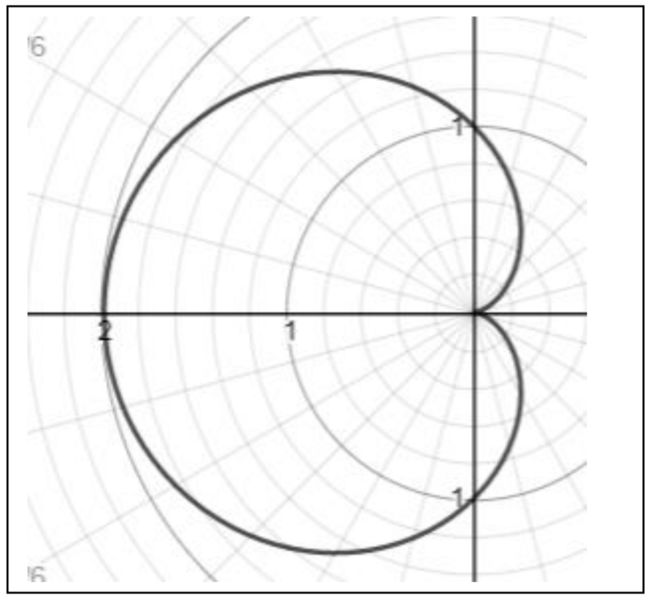
To plot polar curves simply create a table for different θ values and find **r**. This will give you points in the form (r, θ). Plot all of the points.

Example: Sketch the curve of the following polar equations.

a) $r = 1 - \cos(\theta)$

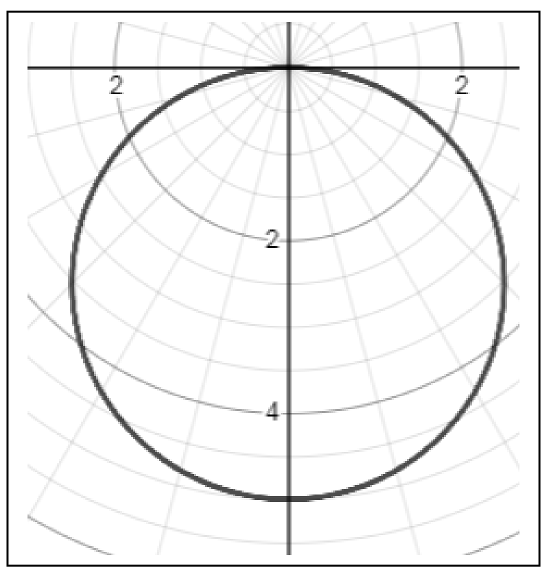
a) solution

r	θ		r	θ
0	0		$\frac{7\pi}{6}$	1.86603
$\frac{\pi}{6}$	.13397		$\frac{5\pi}{4}$	1.70711
$\frac{\pi}{4}$	.29289		$\frac{4\pi}{3}$	1.5
$\frac{\pi}{3}$	.5		$\frac{3\pi}{2}$	1
$\frac{\pi}{2}$	1		$\frac{5\pi}{3}$	.5
$\frac{2\pi}{3}$	1.5		$\frac{7\pi}{4}$	.29289
$\frac{3\pi}{4}$	1.70711		$\frac{11\pi}{6}$	.13397
$\frac{5\pi}{6}$	1.86603		2π	0
π	2			



b) $r = -5\sin(\theta)$

r	θ		r	θ
0	0		$\frac{7\pi}{6}$	2.5
$\frac{\pi}{6}$	-2.5		$\frac{5\pi}{4}$	3.5355
$\frac{\pi}{4}$	-3.5355		$\frac{4\pi}{3}$	4.3301
$\frac{\pi}{3}$	-4.3301		$\frac{3\pi}{2}$	5
$\frac{\pi}{2}$	-5		$\frac{5\pi}{3}$	4.3301
$\frac{2\pi}{3}$	-4.3301		$\frac{7\pi}{4}$	3.5355
$\frac{3\pi}{4}$	-3.5355		$\frac{11\pi}{6}$	2.5
$\frac{5\pi}{6}$	-2.5		2π	0
π	0			



Example: Convert the polar equations from the last example into cartesian equations.

a) $r = 1 - \cos(\theta)$

$r \cdot r = r(1 - \cos(\theta)$ (Multiply both sides by **r**)

$r^2 = r - rcos\,(\theta)$ (Substitute)

$x^2 + y^2 = r - x$ $(r = \sqrt{x^2 + y^2})$

$\boldsymbol{x^2 + y^2 = \sqrt{x^2 + y^2} - x}$

b)) $r = -5\sin(\theta)$

$r \cdot r = -5r\sin(\theta)$ (Multiply both sides by **r**)

$r^2 = -5r\sin(\theta)$

$\boldsymbol{x^2 + y^2 = -5y}$

Tangents to Polar Curves:

To find a tangent line to a polar curve **r = f(Θ),** we let **Θ** be the parameter and write the parametric equations as

$$x = rcos(\theta) = f(\theta)\cos(\theta) \qquad y = rsin(\theta) = f(\theta)\sin(\theta)$$

Then using:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + rcos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - rsin(\theta)}$$

Horizontal tangents occur at $\frac{dy}{d\theta} = 0$. (provided $\frac{dx}{d\theta} \neq 0$)

Vertical tangents occur at $\frac{dx}{d\theta} = 0$.

At the pole, **r = 0**, therefore:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin(\theta) + 0 \cdot cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - 0 \cdot sin(\theta)} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \quad (assuming\ \frac{dr}{d\theta} \neq 0)$$

Example: Find the slope of the tangent line to the given polar curve at the oint specified by Θ.

$$r = 2 + \sin(3\theta), \quad \theta = \frac{\pi}{4}$$

The parametric equations are:

$$x = (2 + \sin(3\theta))\cos(\theta) \qquad y = (2 + \sin(3\theta))\sin(\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos(3\theta) \cdot \sin(\theta) + \cos(\theta) \cdot (2 + \sin(3\theta))}{3\cos(3\theta) \cdot \cos(\theta) - \sin(\theta) \cdot (2 + \sin(3\theta))}$$

Substitute $\theta = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{3\cos(3 \cdot \frac{\pi}{4}) \cdot \sin\left(\frac{\pi}{4}\right) + \cos(\theta) \cdot \left(2 + \sin(3 \cdot \frac{\pi}{4})\right)}{3\cos(3 \cdot \frac{\pi}{4}) \cdot \cos\left(\frac{\pi}{4}\right) - \sin(\frac{\pi}{4}) \cdot \left(2 + \sin(3 \cdot \frac{\pi}{4})\right)}$$

$$\frac{dy}{dx} = \frac{3\left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(2+\frac{\sqrt{2}}{2}\right)}{3\left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(2+\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{\frac{-3\cdot 2}{4} + \sqrt{2} + \frac{2}{4}}{\frac{-3\cdot 2}{4} - \sqrt{2} - \frac{2}{4}} = \frac{\mathbf{-1+\sqrt{2}}}{\mathbf{-2-\sqrt{2}}}$$

To solve for the horizontal and vertical tangents, set $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta} = 0$, respectively and solve for Θ.