7.4 Integration of Rational Functions by Partial Fractions

In order to integrate rational functions (fractions with variables) we must express them as sums of simpler fractions, called **partial fractions**. You may recall that we learned this process in Precalculus. Let's practice this process with the example below.

Example: Find the partial fraction decomposition of

$$\frac{-3x-5}{(x+3)(x-1)}$$

$$\frac{-3x-5}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

Multiply by the Least Common Denominator (LCD) and create a system of equations.

$$\frac{-3x-5}{(x+3)(x-1)} = \frac{A(x-1)}{(x+3)(x-1)} + \frac{B(x+3)}{(x-1)(x+3)} = \frac{Ax - A + Bx + 3B}{(x+3)(x-1)}$$

$$\begin{cases} -3x = Ax + Bx \\ -5 = -A + 3B \end{cases} \Rightarrow \begin{cases} -3 = A + B \\ -5 = -A + 3B \end{cases}$$

Now solve for A and B. By adding the equations A is eliminated and we get:

$$-8 = 4B \;\; so \;\; -2 = B$$

Now we substitute $-2 = B$ in either of the original equations to solve for A.

$$-5 = -A + 3(-2) \;\cdots\; A = -1$$

Therefore:

$$\frac{-3x-5}{(x+3)(x-1)} = \frac{-1}{x+3} + \frac{-2}{x-1}$$

Now let's integrate

Example: Evaluate

$$\int \frac{-3x-5}{(x+3)(x-1)} dx$$

Substituting from the above we get:

$$\int \frac{-3x-5}{(x+3)(x-1)} dx = \int \left(\frac{-1}{x+3} + \frac{-2}{x-1}\right) dx$$

Now integration is much easier.

$$\int \frac{-3x-5}{(x+3)(x-1)} dx = \int \left(\frac{-1}{x+3} + \frac{-2}{x-1}\right) dx = -\ln(x+3) - 2\ln(x-1) + C$$

This can be simplified into one term:

$$-(\ln(x+3) + 2\ln(x-1)) + C \;=\; -(\ln(x+3) + \ln((x-1)^2) + C$$

$$= -\mathbf{ln}((\boldsymbol{x}+\mathbf{3})(\boldsymbol{x}-\mathbf{1})^{\mathbf{2}}) + \boldsymbol{C} \;\; \boldsymbol{or} \;\; \mathbf{ln}\left(\frac{\mathbf{1}}{(\boldsymbol{x}+\mathbf{3})(\boldsymbol{x}-\mathbf{1})^{\mathbf{2}}}\right) + \boldsymbol{C}$$

Example: Evaluate

$$\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx$$

Perform partial fraction decomposition on the integrand.

$$\frac{5x^2 - 3x + 2}{x^3 - 2x^2} = \frac{5x^2 - 3x + 2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$5x^2 - 3x + 2 = A(x)(x-2) + B(x-2) + C(x^2)$$

$$5x^2 = Ax^2 + Cx^2 \qquad -3x = -2Ax + Bx \qquad 2 = -2B$$

Solve for A, B, and C and you get: A = 1, B = - 1, and C = 4. Therefore,

$$\int \frac{5x^2 - 3x + 2}{x^3 - 2x^2} dx = \int \left(\frac{1}{x} + \frac{-1}{x^2} + \frac{4}{x-2}\right) dx = \ln(x) + \frac{1}{x} + 4\ln(x-2) + C$$

$$= \boldsymbol{\frac{1}{x} + \ln(x(x-2)^4) + C}$$

Example: Evaluate

$$\int \left(\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)}\right) dx$$

Find the partial decomposition of the integrand.

$$\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 - 2x + 3}$$

$$7x^2 - 13x + 13 = A(x^2 - 2x + 3) + (Bx + C)(x-2)$$

$$7x^2 - 13x + 13 = Ax^2 - 2Ax + 3A + Bx^2 - 2Bx + Cx - 2C$$

$$7x^2 = Ax^2 + Bx^2 \qquad -13x = -2Ax - 2Bx + Cx \qquad 13 = 3A - 2C$$

Solve for A, B, and C and you get: A = 5, B = 2, and C = 1. Therefore,

$$\int \left(\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)}\right) dx = \int \left(\frac{5}{x-2} + \frac{2x + 1}{x^2 - 2x + 3}\right) dx$$

Let **u** = $x^2 - 2x + 3$ and **du** = $(2x - 2)dx$. For convenience we can write $2x + 1 = (2x - 2) + 3$ thus,

$$\int \frac{2x + 1}{x^2 - 2x + 3} dx = \int \frac{(2x - 2) + 3}{x^2 - 2x + 3} dx = \int \frac{2x - 2}{x^2 - 2x + 3} dx + \int \frac{3}{x^2 - 2x + 3} dx$$

$$\int \left(\frac{7x^2 - 13x + 13}{(x-2)(x^2 - 2x + 3)}\right) dx = \int \frac{5}{x-2} dx + \int \frac{2x - 2}{x^2 - 2x + 3} dx + \int \frac{3}{x^2 - 2x + 3} dx$$

Rewrite $x^2 - 2x + 3$ as $(x-1)^2 + 2$ in the 3rd integral by completing the square and let **u** = $x - 1$ and **du =** dx therefore:

$$\int \left(\frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\right) dx = 5\int \frac{1}{x-2}dx + \int \frac{2x-2}{x^2-2x+3}dx + \int \frac{3}{(x-1)^2+2}dx$$

$$\int \left(\frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\right) dx = 5\ln(x-2) + \ln(x^2-2x+3) + 3\int \frac{1}{u^2+2}dx$$

$$= 5\ln(x-2) + \ln(x^2-2x+3) + \frac{3}{2}\int \frac{1}{\frac{u^2}{2}+1}du$$

$$= 5\ln(x-2) + \ln(x^2-2x+3) + \frac{3}{2}\int \frac{1}{\left(\frac{u}{\sqrt{2}}\right)^2+1}du$$

Using the derivative rule: $\frac{d}{dx}\left[tan^{-1}(x) = \frac{1}{1+x^2}\right] \therefore \int \frac{1}{1+x^2}dx = tan^{-1}x + C$ we can now

$$= 5\ln(x-2) + \ln(x^2-2x+3) + \frac{3}{2}\cdot\sqrt{2}tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

Using the derivative rule: $\frac{d}{dx}\left[tan^{-1}(x) = \frac{1}{1+x^2}\right] \therefore \int \frac{1}{1+x^2}dx = tan^{-1}x + C$ we can now

Remember **u** = x – 1 so when we back substitute we get:

$$\int \left(\frac{7x^2 - 13x + 13}{(x-2)(x^2-2x+3)}\right) dx = \mathbf{5\ln(x-2) + \ln(x^2-2x+3) + \frac{3}{\sqrt{2}}tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C}$$

Example: Evaluate

$$\int \frac{sec^2 t}{tan^2 t + 3\tan t + 2}dt$$

Let **u** = tan(t) and **du** = $sec^2(t)dt$

$$\int \frac{sec^2 t}{tan^2 t + 3\tan t + 2}dt = \int \frac{1}{u^2+3u+2}du$$

Complete the partial fraction decomposition on the integrand.

$$\frac{1}{u^2+3u+2} = \frac{1}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

Solving for A and B we get A = - 1, and B = 1, therefore

$$\int \frac{sec^2 t}{tan^2 t + 3\tan t + 2} dt = \int \frac{1}{u^2+3u+2} du = \int \left(\frac{-1}{u+2} + \frac{1}{u+1}\right) du$$

$$= -\ln(u+2) + \ln(u+1) = \ln\left(\frac{u+1}{u+2}\right) + C$$

$$\int \frac{sec^2 t}{tan^2 t + 3\tan t + 2} dt = \mathbf{\ln\left(\frac{\tan(\boldsymbol{t})+1}{\tan(\boldsymbol{t})+2}\right) + \boldsymbol{C}}$$

Now let's work a problem that requires long division.

Example: Evaluate

$$\int \frac{2x^3+11x^2+28x+33}{x^2-x-6} dx$$

Perform long division and you will get:

$$\frac{2x^3+11x^2+28x+33}{x^2-x-6} = 2x + 13 + \frac{53x+11}{x^2-x-6}$$

Use partial fraction decomposition on the fraction remainder:

$$\frac{53x+11}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

Solve for A and B:

$$53x + 11 = Ax + 2A + Bx - 3B \quad \Rightarrow \quad A = 34 \; and \; B = 19$$

$$\int \frac{2x^3+11x^2+28x+33}{x^2-x-6} dx = \int \left(2x + 13 + \frac{34}{x-3} + \frac{19}{x+2}\right) dx$$

$$\int \frac{2x^3+11x^2+28x+33}{x^2-x-6} dx = \boldsymbol{x^2 + 13x + 34\ln(x-3) + 19\ln(x+2) + C}$$