13.2) Derivatives and Integrals of Vector-Valued Functions

1. Review of Differentiation from Calculus I and II:

Say we have a function $y = f(x)$, a real-valued function of one real variable. Its graph is a curve in the x,y plane that passes the vertical line test. We shall refer to this curve as C.

If C has a nonvertical tangent line at a certain point, the slope of the tangent line is obtained by **differentiation**. Since y is a function of x, we use *ordinary* differentiation, which gives us the **derivative** of y with respect to x, $f'(x)$ or $\frac{dy}{dx}$, in terms of x. Specifically, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$. The quantity $\frac{f(x+h) - f(x)}{h}$ is known as the **difference quotient**. It represents the slope of the **secant line** passing through a fixed point $(x,f(x))$ and a variable point $(x+h,f(x+h))$. As h approaches zero, the latter point approaches the former point and the secant line approaches the tangent line. (Of course, in practice, we find the derivative by using the *rules of differentiation* studied in Calculus I.)

If $f(x)$ is a constant function or a linear function, then C is already a line, so the tangent line coincides with this line itself; hence, the derivative at any point is simply the slope of the original line (which means the derivative is a fixed value–i.e., it does not vary as x varies). For other functions, the derivative is *not* fixed, but rather varies as x varies. For instance, in the case of $y = f(x) = -2x^2 + 5$, the derivative is $-4x$ (so the slope of the tangent line is 12 when x is -3, whereas the slope of the tangent line is -20 when x is 5). In such cases, the derivative is a function of *one* variable, x.

Now suppose the plane curve C is *not* the graph of a function–i.e., the curve does not pass the vertical line test. In practice, C will be the graph of an equation involving x and y (a *relation*) that cannot be solved to give us a unique y for every x. We may be able to graph the equation by hand, as in the case of $x^2 + y^2 = 9$. Or we may need to use a computer to obtain the graph, as in the case of $(x+y)^{x-y} = \ln(x^y - y^x)$.

When y is *not* a function of x, we may find the derivative by using **implicit differentiation**: We differentiate both sides of the equation with respect to x, and then solve the resulting equation for $\frac{dy}{dx}$ in terms of both x and y. For instance, in the case of $x^2 + y^2 = 9$, we get $\frac{dy}{dx} = -\frac{x}{y}$. In such cases, the derivative is a function of *two* variables, x and y. In the case of $x^2 + y^2 = 9$, consider two points that vertically align with each other, $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$. At the former, we get $\frac{dy}{dx} = -1$, and at the latter, we get $\frac{dy}{dx} = 1$. Thus, the equation of the tangent line at the former point is $y = -x + \sqrt{2}$, whereas the equation of the tangent line at the latter point is $y = x - \sqrt{2}$.

When a plane curve has been parameterized, if the curve has a nonvertical tangent line at a certain point, the slope of the tangent line may be obtained by **parametric differentiation**, which gives us the slope in terms of the parameter. You learned this in Calculus II (in particular, Section 10.2 of your text). Suppose our parameter is t. Generically, our parametric equations are $x = x(t)$, $y = y(t)$. Rather than expressing $\frac{dy}{dx}$ as a function of x

(as we would do in the case of ordinary differentiation), or as a function of both x and y (as we would do in the case of implicit differentiation), we instead express $\frac{dy}{dx}$ as a function of t. First, we find the derivatives $\frac{dx}{dt} = x'(t)$ and $\frac{dy}{dt} = y'(t)$. We then divide the latter by the former, and the result is the slope of the tangent line, i.e., $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{y'(t)}{x'(t)}$.

Suppose the circle $x^2 + y^2 = 9$ is parameterized as $x = 3\cos t$, $y = 3\sin t$. Then $\frac{dx}{dt} = -3\sin t$ and $\frac{dy}{dt} = 3\cos t$. When $t = \frac{\pi}{6}$, we get the point $(\frac{3\sqrt{3}}{2}, \frac{3}{2})$, and we get $\frac{dx}{dt} = -\frac{3}{2}$ and $\frac{dy}{dt} = \frac{3\sqrt{3}}{2}$, so $\frac{dy}{dx} = \frac{3\sqrt{3}}{2} \div -\frac{3}{2} = -\sqrt{3}$. Here, we numerically evaluated $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and then divided the latter by the former. Instead, we could set up the ratio $\frac{y'(t)}{x'(t)}$ and then simplify it *before* we evaluate numerically. In this case, we would have $\frac{3\cos t}{-3\sin t}$, which would simplify to $-\cot t$. Evaluating at $\frac{\pi}{6}$ then gives us $-\sqrt{3}$.

By the way, in Calculus I and II, we dealt only with *plane* curves–we did not deal with *space* curves. Now, in Calculus III, we will be dealing with *both* types of curves. In both cases, we will use differentiation to find the tangent line at any point on the curve. However, the concept of *slope* applies only to plane curves, not to space curves (i.e., "slope" is an inherently two-dimensional concept). So when we examine a tangent line to a space curves, we will not discuss its slope (since this would be meaningless). We can and will discuss slopes of tangent lines for plane curves.

2. Differentiation of Vector-Valued Functions:

A vector-valued function of one parameter can be **differentiated** as follows. Assuming the parameter is t, we define the **derivative** with respect to t to be $\lim_{h\to 0} \frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$. This derivative is denoted $\mathbf{r}'(t)$ or $\frac{d}{dt}\mathbf{r}(t)$ or $\frac{d\mathbf{r}}{dt}$ or $D_t\,\mathbf{r}(t)$. We refer to the symbol $\frac{d}{dt}$ or D_t as the **differentiation operator**. Note that the derivative is a vector, not a scalar.

In two dimensions, with $\mathbf{r}(t) = < x(t), y(t) > = x(t)\mathbf{i} + y(t)\mathbf{j}$, we have $\mathbf{r}'(t) = < \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} > = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\mathbf{i} + \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\mathbf{j}$. As a result of the limit, we obtain $\mathbf{r}'(t) = < x'(t), y'(t) > = < \frac{dx}{dt}, \frac{dy}{dt} > = x'(t)\mathbf{i} + y'(t)\mathbf{j} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$.

In three dimensions, with $\mathbf{r}(t) = < x(t), y(t), z(t) > = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, we have $\mathbf{r}'(t) = < \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}, \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} > =$ $\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\mathbf{i} + \lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\mathbf{j} + \frac{z(t+h) - z(t)}{h}\mathbf{k}$. As a result of the limit, we obtain $\mathbf{r}'(t) = < x'(t), y'(t), z'(t) > = < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$.

The differentiation operator distributes over the components of a vector-valued function in the same way that the limit distributes over the components:

- $\frac{d}{dt} < x(t), y(t) > = < \frac{d}{dt}x(t), \frac{d}{dt}y(t) >$
- $\frac{d}{dt} < x(t), y(t), z(t) > = < \frac{d}{dt}x(t), \frac{d}{dt}y(t), \frac{d}{dt}z(t) >$

Differentiation Rules:

- $\frac{d}{dt}\mathbf{C} = \mathbf{0}$, where $\mathbf{C}$ is any constant vector.
- $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{w}(t)] = \mathbf{u}'(t) + \mathbf{w}'(t)$ The Addition Rule
- $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{w}(t)] = \mathbf{u}'(t) - \mathbf{w}'(t)$ The Subtraction Rule
- $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$ The Constant Factor Rule
- $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ Product Rule #1
- $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{w}(t)] = \mathbf{u}'(t) \cdot \mathbf{w}(t) + \mathbf{u}(t) \cdot \mathbf{w}'(t)$ Product Rule #2
- $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{w}(t)] = \mathbf{u}'(t) \times \mathbf{w}(t) + \mathbf{u}(t) \times \mathbf{w}'(t)$ Product Rule #3
- $\frac{d}{dt}[\mathbf{u}(g(t))] = \mathbf{u}'(g(t))\, g'(t)$, normally written $g'(t)\mathbf{u}'(g(t))$ The Chain Rule

When $\mathbf{r}(t)$ represents a *position function* and the parameter t represents *time*, then $\mathbf{r}'(t)$ is interpreted as the **velocity function**, in which case we may write $\mathbf{v}(t)$ in place of $\mathbf{r}'(t)$. (In this context, the curve represented by $\mathbf{r}(t)$ may be referred to as the *curve of motion* or the *path of motion*.)

Notice that $\mathbf{v}(t)$ is a vector. The *magnitude* of velocity is the **speed of motion**, $v(t) = |\mathbf{v}(t)|$. We normally just call this the "speed." (However, there is another kind of speed that we will discuss later, so sometimes we need to use the complete phrase "speed of motion" to clarify that we mean the magnitude of velocity.)

- For two-dimensional motion, $v(t) = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.
- For three-dimensional motion, $v(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} =$ $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$.

Velocity is a vector and speed is a scalar. If we say that "velocity and speed are zero," we mean that velocity is the *zero vector*, $\mathbf{0}$, and speed is the *real number zero*, 0. If we say that "velocity and speed are nonzero," we mean that velocity is a *nonzero vector* and speed is a *nonzero real number*. (Of course, velocity and speed must be either both zero or both nonzero–we cannot have one being zero without the other being zero.)

If $\mathbf{v}(t)$ is nonzero, then it has a direction, which represents the *instantaneous direction of motion.* This direction will be along the curve's tangent line at the given point. (To write the equation of the tangent line, we use the velocity vector as the line's direction vector.) On the other hand, if $\mathbf{v}(t) = \mathbf{0}$, then it has no direction, so the instantaneous direction of motion is *undefined*.

At any instant where velocity is nonzero, the curve will have a tangent line; the curve is said to be **smooth** at such a point. At any instant where velocity is zero, the curve will have a **cusp** (also known as a *kink* or a *sharp turn*); the curve is *not smooth* at such a point. When there is a cusp, the curve may or may not have a tangent line (depending on whether or not the *left-hand tangent* coincides with the *right-hand tangent*).

For example, the plane curve $y^3 = x^2$ or $y = x^{2/3}$ can be parameterized as $x = t^3$, $y = t^2$. With this parameterization, we have $\mathbf{v}(t) = < 3t^2, 2t >$. Since $\mathbf{v}(0) = \mathbf{0}$, the curve has a cusp when $t = 0$, i.e., at the point $(0,0)$. The curve has a vertical tangent line at this point. (Note: In Calculus I, if we differentiated the equation $y = x^{2/3}$, we would get $\frac{dy}{dx} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$, which is undefined when x is zero; this makes sense, because a vertical tangent line has undefined slope.) The curve is smooth at every point *other than* the origin.

By the way, a curve can have a vertical tangent line even when it is smooth (in other words, you don't need a cusp to have a vertical tangent line). A circle, for instance, has two vertical tangent lines, but it is smooth at every point (i.e., it has no cusps).

The plane curve $y^2 = x^3$ or $y = \pm x^{3/2}$ can be parameterized as $x = t^2$, $y = t^3$. With this parameterization, we have $\mathbf{v}(t) = < 2t, 3t^2 >$. Since $\mathbf{v}(0) = \mathbf{0}$, the curve has a cusp when $t = 0$, i.e., at the point $(0,0)$. The curve has a horizontal tangent line at this point. (Note: In Calculus I, if we differentiated the equation $y = \pm x^{3/2}$, we would get $\frac{dy}{dx} = \pm\frac{3}{2}x^{1/2}$, which gives us $\frac{dy}{dx} = 0$ when x is zero. On the other hand, if we *implicitly* differentiated the equation $y^2 = x^3$, we'd get $2y\frac{dy}{dx} = 3x^2$, which would give us $\frac{dy}{dx} = \frac{3x^2}{2y}$ *provided* y is nonzero.)

For the parabola $x = t,\ y = -2t^2 + 5$, we have position function $\mathbf{r}(t) = < t, -2t^2 + 5 >$, velocity function $\mathbf{v}(t) = < 1, -4t >$, and speed function $v(t) = \sqrt{1 + 16t^2}$. When $t = 3$, we have the point $(3, -13)$, the position vector $< 3, -13 >$, the velocity vector $< 1, -12 >$, and speed $\sqrt{145} \approx 12.04$. At the point $(3, -13)$, the tangent line has parametric equations $x = 3 + t$, $y = -13 - 12t$.

For the helix $x = 3\cos t$, $y = 3\sin t$, $z = t$, we have position function $\mathbf{r}(t) = < 3\cos t, 3\sin t, t >$, velocity function $\mathbf{v}(t) = < -3\sin t, 3\cos t, 1 >$, and speed function $v(t) = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$. Notice that in this case we have constant speed. (Bear in mind, there are many possible parameterizations of the curve, some of which might *not* have constant speed.) When $t = \pi$, we have the point $(-3, 0, \pi)$, the position vector $< -3, 0, \pi >$, the velocity vector $< 0, -3, 1 >$, and speed $\sqrt{10} \approx 3.162$. At the point $(-3, 0, \pi)$, the tangent line has parametric equations $x = -3 + 0t$, $y = 0 - 3t$, $z = \pi + 1t$, in other words, $x = -3$, $y = -3t$, $z = \pi + t$.

In the above example, the speed is constant, but the velocity is *not* constant. Whereas speed is a scalar, velocity is a vector, and as such it has both magnitude and direction (except when it is zero, in which case it has no direction). The magnitude of the velocity was fixed at $\sqrt{10}$, but the direction of the velocity was *not* fixed–it changes from instant to instant.

In many applications, we are interested only in the *direction* of motion, and not in the *speed* of motion. In such cases, we do not care how fast our particle is moving, but we do care about the particle's direction at any instant. We may also be interested in how the direction is changing (without any regard to how the speed may be changing). In these situations, we do not focus on the velocity vector. Instead, we focus on a unit vector (i.e., a vector of length one) having the same direction as velocity. We call this vector the **unit tangent vector**, and we denote it $\mathbf{T}(t)$. Of course, we have $\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) = \frac{\mathbf{v}(t)}{v(t)}$. Note that $\mathbf{T}(t)$ is undefined whenever $v(t) = 0$.

- In two dimensions, $\mathbf{T}(t) = (x'(t)^2 + y'(t)^2)^{-1/2} < x'(t), y'(t) >$.
- In three dimensions, $\mathbf{T}(t) = (x'(t)^2 + y'(t)^2 + z'(t)^2)^{-1/2} < x'(t), y'(t), z'(t) >$.

In the case of the parabola discussed above, $\mathbf{T}(t) = (1 + 16t^2)^{-1/2} < 1, -4t >$ or $< \frac{1}{\sqrt{1+16t^2}}, \frac{-4t}{\sqrt{1+16t^2}} >$.

In the case of the helix discussed above, $\mathbf{T}(t) = \frac{1}{\sqrt{10}} < -3\sin t, 3\cos t, 1 >$ or $< \frac{-3\sin t}{\sqrt{10}}, \frac{3\cos t}{\sqrt{10}}, \frac{1}{\sqrt{10}} >$

If we are interested in how the direction is *changing*, we would need to differentiate $\mathbf{T}(t)$ with respect to time, i.e., we would need to find $\frac{d}{dt}\mathbf{T}(t) = \mathbf{T}'(t) = \frac{d\mathbf{T}}{dt}$. This is a rather tricky topic; we will postpone discussing it for the moment.

On the other hand, it is quite straightforward to consider how *velocity* is changing. We simply differentiate $\mathbf{v}(t)$ with respect to time. Since $\mathbf{v}(t) = \mathbf{r}'(t)$, $\mathbf{v}'(t) = \mathbf{r}''(t)$, the *second derivative* of the position function (which could also be expressed as $\frac{d^2\mathbf{r}}{dt^2}$). We call this the **acceleration function**, and we denote it $\mathbf{a}(t)$.

- In two dimensions, $\mathbf{a}(t) = < x''(t), y''(t) > = < \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} > = x''(t)\mathbf{i} + y''(t)\mathbf{j} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$.
- In three dimensions, $\mathbf{a}(t) = < x''(t), y''(t), z''(t) > = < \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} >$ $= x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$.

Since velocity is a vector that shows both the speed of motion and the direction of motion, and since acceleration is the *rate of change* of velocity, acceleration reflects *both* how the speed of motion is changing *and* how the direction is changing. In constrast, the unit tangent vector shows *only* the direction of motion, so its rate of change, $\mathbf{T}'(t)$, reflects *only* how the direction is changing. (In other words, $\mathbf{T}(t)$ tells us nothing about the speed of motion, so $\mathbf{T}'(t)$ tells us nothing about how the speed of motion is changing.)

The **magnitude of acceleration** is $a(t) = |\mathbf{a}(t)|$.

- In two dimensions, $a(t) = \sqrt{x''(t)^2 + y''(t)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$.
- In two dimensions, $a(t) = \sqrt{x''(t)^2 + y''(t)^2 + z''(t)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$.

In the case of the parabola discussed above, $\mathbf{a}(t) = < 0, -4 >$ and $a(t) = 4$. This is a case of constant acceleration.

In the case of the helix discussed above, $\mathbf{a}(t) = < -3\cos t, -3\sin t, 0 >$ and $a(t) = \sqrt{9\cos^2 t + 9\sin^2 t} = 3$. In this case, the magnitude of acceleration is constant, but acceleration itself is not constant (its direction is changing).

For two-dimensional motion, suppose we have a value of t where $x'(t) = 0$ and $y'(t) \neq 0$. Then $\mathbf{v}(t)$ will be a nonzero scalar multiple of $\mathbf{j}$ and hence the curve will have a **vertical tangent line** at the point in question. On the other hand, if we have a value of t where $y'(t) = 0$ and $x'(t) \neq 0$, then $\mathbf{v}(t)$ will be a nonzero scalar multiple of $\mathbf{i}$ and the curve has a **horizontal tangent line** at the point in question. If $x'(t)$ and $y'(t)$ are both nonzero, the curve has an **oblique** (**or slanted**) **tangent line** at the point in question. If $x'(t) = 0$ and $y'(t) = 0$, then the curve has a cusp at the point in question. At this point, there may or may not be a tangent line; if there is a tangent line, it could be vertical or horizontal or oblique. At any point where the curve has a horizontal or oblique tangent line, the tangent line has a slope, which is $\frac{dy}{dx}$. If the curve is smooth at this point, then $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. This equation is not applicable at a cusp. For instance, we saw earlier that the curve $x = t^2$, $y = t^3$ has a horizontal tangent line when $t = 0$, i.e., at the point $(0,0)$, so $\frac{dy}{dx} = 0$ at that point, but this point is a cusp, so the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ does not apply (if we tried to apply it, the result would be undefined).

At any smooth point, $\frac{dy}{dx}$ is positive and the tangent line is rising when $x'(t)$ and $y'(t)$ are both positive or both negative; $\frac{dy}{dx}$ is negative and the tangent line is falling when $x'(t)$ is positive and $y'(t)$ is negative or vice versa.

For the circle $x = 3\cos t$, $y = 3\sin t$, we have $\mathbf{v}(t) = < -3\sin t, 3\cos t >$. $\mathbf{v}(0) = < 0, 3 > = 3\mathbf{j}$ and $\mathbf{v}(\pi) = < 0, -3 > = -3\mathbf{j}$, so the circle has vertical tangent lines when $t = 0$ and $t = \pi$, i.e., at the points $(3,0)$ and $(-3,0)$. $\mathbf{v}(\frac{\pi}{2}) = < -3, 0 > = -3\mathbf{i}$ and $\mathbf{v}(\frac{3\pi}{2}) = < 3, 0 > = 3\mathbf{i}$, so the circle has horizontal tangent lines when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$, i.e., at the points $(0,3)$ and $(0,-3)$. $\mathbf{v}(t)$ is never zero because the sine and cosine functions are never simultaneously zero; hence the circle has no cusps.

The Orthogonal Derivative Theorem: Any vector-valued function with constant magnitude is always orthogonal to its own derivative.

- For a position function $\mathbf{r}(t)$, if $r(t)$ is constant (i.e., if our moving particle has a fixed distance from the origin), then $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal. This applies to motion upon a circle or sphere centered at the origin.
- For a velocity function $\mathbf{v}(t)$, if $v(t)$ is constant (i.e., if our moving particle has a fixed speed), then $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal.
- By definition, $\mathbf{T}(t)$ has constant magnitude (because it is a unit vector). Thus, $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal.

Proof:
Let $\mathbf{r}(t)$ be a vector-valued function. Suppose $r(t) = c$ for all t. Then $r(t)^2 = c^2$ for all t.
$r(t)^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$, so $\mathbf{r}(t) \cdot \mathbf{r}(t) = c^2$ for all t.
Differentiate both sides of this equation with respect to t...
$\frac{d}{dt}\, \mathbf{r}(t) \cdot \mathbf{r}(t) = \frac{d}{dt}\, c^2$
The right side of this equation is 0. For the left side, we apply the Product Rule...
$\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$
$\mathbf{r}(t) \cdot \mathbf{r}'(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$
$2\, \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$

$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$
Hence, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.
QED.

For the circle $x = 3\cos t,\ y = 3\sin t,$ we have $\mathbf{r}(t) = < 3\cos t, 3\sin t >,\ \mathbf{v}(t) = < -3\sin t, 3\cos t >$ and $\mathbf{a}(t) = < -3\cos t, -3\sin t >$. $\mathbf{r}(t) \cdot \mathbf{v}(t) = -9\cos t \sin t + 9\sin t \cos t = 0,$ and $\mathbf{v}(t) \cdot \mathbf{a}(t) = 9\sin t \cos t - 9\cos t \sin t = 0.$ This confirms the Orthogonal Derivative Theorem. (Notice that with this parameterization, the speed of motion is fixed: $v(t) = 3$ for all t.)

3. **Integration of Vector-Valued Functions**:

Given two vector-valued functions $\mathbf{u}(t)$ and $\mathbf{w}(t)$, if $\mathbf{u}'(t) = \mathbf{w}(t)$ for all t in an open interval, then $\mathbf{w}(t)$ is the **derivative** of $\mathbf{u}(t)$, and $\mathbf{u}(t)$ is an **antiderivative** of $\mathbf{w}(t)$. (An antiderivative of $\mathbf{w}(t)$ is a function whose derivative is $\mathbf{w}(t)$. In other words, it is a function that you can differentiate to obtain $\mathbf{w}(t)$.)

For example, consider $\mathbf{u}(t) = < t^2, \sin t >$ and $\mathbf{w}(t) = < 2t, \cos t >$. $\mathbf{u}'(t) = \mathbf{w}(t)$ for all $t \in (-\infty, \infty)$, so $\mathbf{w}(t)$ is the derivative of $\mathbf{u}(t)$, and $\mathbf{u}(t)$ is an antiderivative of $\mathbf{w}(t)$.

Pay close attention to the wording used above. We say $\mathbf{w}(t)$ is "the" derivative of $\mathbf{u}(t)$ because $\mathbf{u}(t)$ has a *unique* derivative, but we say $\mathbf{u}(t)$ is "an" antiderivative of $\mathbf{w}(t)$ because $\mathbf{w}(t)$ will have *infinitely many* antiderivatives. For any constant vector $\mathbf{C}$, the function $\mathbf{u}(t) + \mathbf{C}$ is an antiderivative of $\mathbf{w}(t)$, because $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{C}) = \frac{d}{dt}\mathbf{u}(t) + \frac{d}{dt}\mathbf{C} = \mathbf{u}'(t) + \mathbf{0} = \mathbf{u}'(t) = \mathbf{w}(t)$. The collection of *all* antiderivatives of $\mathbf{w}(t)$ is called the **indefinite integral** of $\mathbf{w}(t)$ and is denoted $\int \mathbf{w}(t)\, dt$. We may write $\int \mathbf{w}(t)\, dt = \mathbf{u}(t) + \mathbf{C}$, where $\mathbf{C}$ is an **arbitrary constant vector**. The indefinite integral of $\mathbf{w}(t)$ can also be referred to as the **general antiderivative** of $\mathbf{w}(t)$.

In the above example, the general antiderivative of $< 2t, \cos t >$ is $< t^2, \sin t > + \mathbf{C}$.

In two-dimensional space, $\mathbf{C}$ can be expressed as $< C_1, C_2 >$. In three-dimensional space, it can be expressed as $< C_1, C_2, C_3 >$. Thus, in the above example, we can write the general antiderivative of $< 2t, \cos t >$ as $< t^2, \sin t > + < C_1, C_2 >$, or as $< t^2 + C_1, \sin t + C_2 >$.

- In two-dimensional space, if $\mathbf{w}(t) = < x(t), y(t) >$, then $\int \mathbf{w}(t)\, dt = \int < x(t), y(t) > dt = < \int x(t)\, dt, \int y(t)\, dt >$. We could also write $\int \mathbf{w}(t)\, dt = \int (x(t)\mathbf{i} + y(t)\mathbf{j})\, dt = \int x(t)\, dt\, \mathbf{i} + \int y(t)\, dt\, \mathbf{j}$.
- In three-dimensional space, if $\mathbf{w}(t) = < x(t), y(t), z(t) >$, then $\int \mathbf{w}(t)\, dt = \int < x(t), y(t), z(t) > dt = < \int x(t)\, dt, \int y(t)\, dt, \int z(t)\, dt >$. We could also write $\int \mathbf{w}(t)\, dt = \int (x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})\, dt = \int x(t)\, dt\, \mathbf{i} + \int y(t)\, dt\, \mathbf{j} + \int z(t)\, dt\, \mathbf{k}$.

Thus, the integration operator distributes over the components of a vector-valued function, just like the differentiation operator. When we integrate each component, we obtain an arbitrary constant for each one; be sure to use subscripts for the arbitrary constants, since if you just wrote C for each one, you'd be implying all the constants are the same, which is not generally the case. Alternatively, you could separate out the constants into the single vector term $\mathbf{C}$.

For example, $\int < t^3, \frac{1}{t}, \sec^2 t > dt = < \int t^3 dt, \int \frac{1}{t} dt, \int \sec^2 t\, dt > =$
$< \frac{1}{4}t^4 + C_1, \ln|t| + C_2, \tan t + C_3 >$, or $< \frac{1}{4}t^4, \ln|t|, \tan t > + \mathbf{C}$.

A generic antiderivative of $\mathbf{w}(t)$ can be denoted $\mathbf{W}(t)$.

A **particular antiderivative** can be dictated by an **initial condition**. For instance, suppose we seek the antiderivative of $\mathbf{w}(t) = < 2t, \cos t >$ whose value when $t = \frac{\pi}{2}$ is $< 5, 7 >$. In other words, find $\mathbf{W}(t)$ so that $\mathbf{W}(\frac{\pi}{2}) = < 5, 7 >$. We already know that the general antiderivative of $< 2t, \cos t >$ is $< t^2 + C_1, \sin t + C_2 >$. Hence, the challenge is to find the necessary values of the constants C_1 and C_2. $\left(\frac{\pi}{2}\right)^2 + C_1 = 5$, so $C_1 = 5 - \frac{\pi^2}{4} = \frac{20-\pi^2}{4}$, and $\sin\frac{\pi}{2} + C_2 = 7$, so $C_2 = 6$. Hence, we want the particular antiderivative $\mathbf{W}(t) =$
$< t^2 + \frac{20-\pi^2}{4}, \sin t + 6 >$. We could also write this as $< t^2, \sin t > + < \frac{20-\pi^2}{4}, 6 >$.

Here is a physics application: Suppose a particle is moving through space with acceleration $\mathbf{a}(t) = < 12t^2 + 2, \frac{3}{4\sqrt{t}}, 50e^{5t} >$. At $t = 1$, its position is $< 10, 1, 2e^5 + 13 >$ and its velocity is $< 11, \frac{3}{2}, 10e^5 + 4 >$. Let us find its position and velocity functions. First, we will find the velocity function by integrating the acceleration function. $\int < 12t^2 + 2, \frac{3}{4\sqrt{t}}, 50e^{5t} > dt =$
$< 4t^3 + 2t + C_1, \frac{3}{2}\sqrt{t} + C_2, 10e^{5t} + C_3 >$. When $t = 1$, we get $< 6 + C_1, \frac{3}{2} + C_2, 10e^5 + C_3 > =$
$< 11, \frac{3}{2}, 10e^5 + 4 >$, so $C_1 = 5$, $C_2 = 0$, and $C_3 = 4$. Thus, our velocity function is $\mathbf{v}(t) =$
$< 4t^3 + 2t + 5, \frac{3}{2}\sqrt{t}, 10e^{5t} + 4 >$. Next, we will find the position function by integrating the velocity function. $\int < 4t^3 + 2t + 5, \frac{3}{2}\sqrt{t}, 10e^{5t} + 4 > dt =$
$< t^4 + t^2 + 5t + D_1, t^{3/2} + D_2, 2e^{5t} + 4t + D_3 >$. When $t = 1$, we get
$< 7 + D_1, 1 + D_2, 2e^5 + 4 + D_3 > = < 10, 1, 2e^5 + 13 >$, so $D_1 = 3$, $D_2 = 0$, and $D_3 = 9$. Thus, our position function is $\mathbf{r}(t) = < t^4 + t^2 + 5t + 3, t^{3/2}, 2e^{5t} + 4t + 9 >$.

For any real numbers a and b, $\int_a^b \mathbf{w}(t)\, dt$ is known as the **definite integral** of $\mathbf{w}(t)$ over the interval (on the t axis) with endpoints a and b. a and b are known as the **limits or boundaries of integration**. Whereas the indefinite integral of $\mathbf{w}(t)$ gives us an infinite collection of vector-valued functions, the definite integral of $\mathbf{w}(t)$ gives us a particular vector (rather than a vector-valued function). Note the similarity to what you learned in Calculus I: If $f(x)$ is a real-valued function, then the indefinite integral of $f(x)$ gives an infinite collection of real-valued functions, whereas the definite integral of $f(x)$ gives us a particular real number (rather than a real-valued function).

- In two-dimensional space, if $\mathbf{w}(t) = < x(t), y(t) >$, then $\int_a^b \mathbf{w}(t)\ dt = \int_a^b < x(t), y(t) > dt =$ $< \int_a^b x(t)\ dt, \int_a^b y(t)\ dt >$. We could also write $\int_a^b \mathbf{w}(t)\ dt = \int_a^b (x(t)\mathbf{i} + y(t)\mathbf{j})\ dt =$ $\int_a^b x(t)\ dt\ \mathbf{i} + \int_a^b y(t)\ dt\ \mathbf{j}$.

- In three-dimensional space, if $\mathbf{w}(t) = < x(t), y(t), z(t) >$, then $\int_a^b \mathbf{w}(t)\ dt =$ $\int_a^b < x(t), y(t), z(t) > dt = < \int_a^b x(t)\ dt, \int_a^b y(t)\ dt, \int_a^b z(t)\ dt >$. We could also write $\int_a^b \mathbf{w}(t)\ dt = \int_a^b (x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k})\ dt = \int_a^b x(t)\ dt\ \mathbf{i} + \int_a^b y(t)\ dt\ \mathbf{j} + \int_a^b z(t)\ dt\ \mathbf{k}$.

Thus, the definite integral distributes over the components of a vector-valued function, just like the indefinite integral.

We can adapt the Fundamental Theorem of Calculus to our study of vectors: If $\mathbf{W}(t)$ is any antiderivative of $\mathbf{w}(t)$, then $\int_a^b \mathbf{w}(t)\ dt = \mathbf{W}(b) - \mathbf{W}(a)$. This may be denoted $[\mathbf{W}(t)]_a^b$

We have already seen that $< t^2, \sin t >$ is an antiderivative of $< 2t, \cos t >$. Hence,
$\int_{\pi/4}^{\pi/2} < 2t, \cos t > dt = [< t^2, \sin t >]_{\pi/4}^{\pi/2} = < \frac{\pi^2}{4}, 1 > - < \frac{\pi^2}{16}, \frac{\sqrt{2}}{2} > = < \frac{\pi^2}{4} - \frac{\pi^2}{16}, 1 - \frac{\sqrt{2}}{2} > =$
$< \frac{3\pi^2}{16}, \frac{2-\sqrt{2}}{2} >$.

4. Further Discussion of the Unit Tangent Vector's Derivative:

$\mathbf{T}(t)$ is a vector-valued function of time t, but it is always a vector with a *fixed* length (namely, length one). Hence, as time varies, the only thing about $\mathbf{T}(t)$ that can change is its *direction*. The *rate* at which our particle changes direction is found by differentiating $\mathbf{T}(t)$ with respect to time, in other words, by finding $\mathbf{T}'(t)$. But bear in mind, the derivative of a vector-valued function is another vector-valued function. If we wish to express the rate of direction change as a *scalar*, then we compute the *magnitude* of $\mathbf{T}'(t)$.

$|\mathbf{T}'(t)|$ can be thought of as the **speed of direction change**. In contrast, $v(t) = |\mathbf{v}(t)|$ is the **speed of motion**. Whenever we speak of "speed," if we don't specify which kind of speed we mean, then we implicitly mean *speed of motion*.

Since $\mathbf{T}(t)$ is undefined when $v(t) = 0$, $\mathbf{T}'(t)$ and $|\mathbf{T}'(t)|$ are likewise undefined when $v(t) = 0$. Thus, in the following discussion, we assume $v(t)$ is nonzero.

Theorem 1: $\mathbf{T}'(t) = v(t)^{-3}\left[v(t)^2\mathbf{a}(t) - \mathbf{v}(t) \cdot \mathbf{a}(t)\ \mathbf{v}(t)\right] = \frac{v(t)^2\mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)}{v(t)^3}$.

Proof:

Since $\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t)$, to differentiate $\mathbf{T}(t)$, we apply the Product Rule, giving us $\mathbf{T}'(t) = \left(\frac{1}{v(t)}\right)'\mathbf{v}(t) + \frac{1}{v(t)}\mathbf{v}'(t) = \left(\frac{1}{v(t)}\right)'\mathbf{v}(t) + \frac{1}{v(t)}\mathbf{a}(t)$.

$\frac{1}{v(t)} = (x'(t)^2 + y'(t)^2)^{-1/2}$ in two dimensions, and $\frac{1}{v(t)} = (x'(t)^2 + y'(t)^2 + z'(t)^2)^{-1/2}$ in three dimensions. To differentiate $\frac{1}{v(t)}$, we apply the Chain Rule. In three dimensions, $\left(\frac{1}{v(t)}\right)' = -\frac{1}{2}(x'(t)^2 + y'(t)^2 + z'(t)^2)^{-3/2}(2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t)) =$
$-(x'(t)^2 + y'(t)^2 + z'(t)^2)^{-3/2}(x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)) =$
$-v(t)^{-3}\ \mathbf{v}(t) \cdot \mathbf{a}(t)$. In two dimensions, we get the same result.

Now we have $\mathbf{T}'(t) = -v(t)^{-3}\ \mathbf{v}(t) \cdot \mathbf{a}(t)\ \mathbf{v}(t) + v(t)^{-1}\mathbf{a}(t)$
$= v(t)^{-1}\mathbf{a}(t) - v(t)^{-3}\ \mathbf{v}(t) \cdot \mathbf{a}(t)\ \mathbf{v}(t)$
$= v(t)^{-3}\left[v(t)^2\mathbf{a}(t) - \mathbf{v}(t) \cdot \mathbf{a}(t)\ \mathbf{v}(t)\right]$
$= \frac{v(t)^2\mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)}{v(t)^3}$.

QED.

We could also write $\mathbf{T}'(t)$ as $v(t)^{-3}\left[\mathbf{v}(t) \cdot \mathbf{v}(t)\ \mathbf{a}(t) - \mathbf{v}(t) \cdot \mathbf{a}(t)\ \mathbf{v}(t)\right]$ or $\frac{\mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)}{v(t)^3}$.

For brevity, we may write $\mathbf{T}' = v^{-3}\left[v^2\mathbf{a} - \mathbf{v} \cdot \mathbf{a}\ \mathbf{v}\right] = \frac{v^2\mathbf{a} - \mathbf{v}\cdot\mathbf{a}\ \mathbf{v}}{v^3} =$
$v^{-3}\left[\mathbf{v} \cdot \mathbf{v}\ \mathbf{a} - \mathbf{v} \cdot \mathbf{a}\ \mathbf{v}\right] = \frac{\mathbf{v}\cdot\mathbf{v}\ \mathbf{a} - \mathbf{v}\cdot\mathbf{a}\ \mathbf{v}}{v^3}$.

It follows that $|\mathbf{T}'(t)| = \left|\frac{\mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)}{v(t)^3}\right| = \frac{|\mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)|}{v(t)^3}$.

For brevity, we may write $|\mathbf{T}'| = \frac{|\mathbf{v}\cdot\mathbf{v}\ \mathbf{a} - \mathbf{v}\cdot\mathbf{a}\ \mathbf{v}|}{v^3}$.

Theorem 2: $\mathbf{v}(t) \times \mathbf{a}(t) = v(t)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$

The proof of Theorem 2 will be postponed until the end of this section.

For brevity, we may write $\mathbf{v} \times \mathbf{a} = v^2(\mathbf{T} \times \mathbf{T}')$.

Theorem 2 can be used in two-dimensional space if we assign a third component of 0 to our position, velocity, and acceleration functions.

Actually, we don't really apply Theorem 2 in practice. The only significance of Theorem 2 is that we use it in proving the following theorem (which *is* useful in practice)...

Theorem 3: $|\mathbf{T}'(t)| = v(t)^{-2}|\mathbf{v}(t) \times \mathbf{a}(t)| = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^2}$

Proof:
By Theorem 2, $\mathbf{v}(t) \times \mathbf{a}(t) = v(t)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$.
Therefore $|\mathbf{v}(t) \times \mathbf{a}(t)| = v(t)^2|\mathbf{T}(t) \times \mathbf{T}'(t)|$.
$|\mathbf{T}(t) \times \mathbf{T}'(t)| = |\mathbf{T}(t)||\mathbf{T}'(t)|\sin\frac{\pi}{2} = |\mathbf{T}'(t)|$. (Here we use Theorem 3 from Section 12.4.)
Hence, $|\mathbf{v}(t) \times \mathbf{a}(t)| = v(t)^2|\mathbf{T}'(t)|$
It follows that $|\mathbf{T}'(t)| = v(t)^{-2}|\mathbf{v}(t) \times \mathbf{a}(t)| = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^2}$.
QED.

For brevity, we may write $|\mathbf{T}'| = v^{-2}|\mathbf{v} \times \mathbf{a}| = \frac{|\mathbf{v} \times \mathbf{a}|}{v^2}$.

We now have two formulas for $|\mathbf{T}'|$, namely, $\frac{|\mathbf{v}\cdot\mathbf{v}\,\mathbf{a} - \mathbf{v}\cdot\mathbf{a}\,\mathbf{v}|}{v^3}$ and $\frac{|\mathbf{v} \times \mathbf{a}|}{v^2}$. You may use whichever you prefer, but you'll often find the second more convenient. In either case, you can find $|\mathbf{T}'|$ once you know $\mathbf{v}$, $\mathbf{a}$, and v.

CAUTION: Theorem 3 does *not* imply that $\mathbf{T}'(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{v(t)^2}$. This is *not* a correct equation!

Theorem 3, like Theorem 2, can be used in two-dimensional space if we assign a third component of 0 to our position, velocity, and acceleration functions. This will be illustrated in the following example.

Suppose a particle is moving along the parabola $y = x^2$ with position function $\mathbf{r}(t) = < t, t^2 >$.
Then:

- $\mathbf{v}(t) = < 1, 2t >$
- $v(t) = \sqrt{1 + 4t^2} = (1 + 4t^2)^{1/2}$
- $\mathbf{v}(t) \cdot \mathbf{v}(t) = v(t)^2 = 1 + 4t^2$
- $v(t)^3 = (1 + 4t^2)^{3/2}$ and $v(t)^{-3} = (1 + 4t^2)^{-3/2}$
- $\mathbf{T}(t) = (1 + 4t^2)^{-1/2} < 1, 2t > = \frac{<1,2t>}{\sqrt{1+4t^2}}$
- $\mathbf{a}(t) = < 0, 2 >$
- $\mathbf{v}(t) \cdot \mathbf{a}(t) = (1)(0) + (2t)(2) = 4t$
- $\mathbf{T}'(t) = \frac{(1 + 4t^2)\mathbf{a} - 4t\mathbf{v}}{(1 + 4t^2)^{3/2}}$ by Theorem 1.

  The numerator is $(1 + 4t^2) < 0, 2 > - 4t < 1, 2t > =$
  $< 0, 2 + 8t^2 > - < 4t, 8t^2 > = < -4t, 2 > = 2 < -2t, 1 >$,
  so $\mathbf{T}'(t) = \frac{2<-2t,1>}{(1 + 4t^2)^{3/2}}$ or $\frac{2}{(1 + 4t^2)^{3/2}} < -2t, 1 >$, or $2(1 + 4t^2)^{-3/2} < -2t, 1 >$

- $|\mathbf{T}'(t)| = 2(1+4t^2)^{-3/2}|< -2t, 1 >| = 2(1+4t^2)^{-3/2}\sqrt{4t^2+1} = 2(1+4t^2)^{-3/2}(1+4t^2)^{1/2} = 2(1+4t^2)^{-1} = \frac{2}{1+4t^2}$.
  (We got this directly, without using Theorem 3.)
- $\mathbf{v}(t) \times \mathbf{a}(t) = < 1, 2t, 0 > \times < 0, 2, 0 > = < 0, 0, 2 >$
  (We got this directly. We could have used Theorem 2, but it would be very messy.)
- $|\mathbf{v}(t) \times \mathbf{a}(t)| = 2$
- $|\mathbf{T}'(t)| = \frac{2}{1+4t^2}$ by Theorem 3.

The formula $\mathbf{T}' = \frac{\mathbf{v}\cdot\mathbf{v}\ \mathbf{a} - \mathbf{v}\cdot\mathbf{a}\ \mathbf{v}}{v^3}$ gives us $\mathbf{T}'$ in terms of $\mathbf{v}$, v, and $\mathbf{a}$. In the above example, we were able to write the result entirely in terms of t and simplify down to a nice, clean formula. This is not always possible, or it may be prohibitively difficult. In some cases, we may get $\mathbf{v}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{a}$, and v^3 in terms of t, and then leave $\mathbf{T}'$ in terms of t, $\mathbf{v}$, and $\mathbf{a}$. If we had done so in the above example, we would have left $\mathbf{T}'$ written as $\frac{(1+4t^2)\mathbf{a} - 4t\mathbf{v}}{(1+4t^2)^{3/2}}$.

All these functions may be evaluated at any given value of t. For example, when $t = 3$, we get:

- $\mathbf{r}(3) = < 3, 9 >$
- $\mathbf{v}(3) = < 1, 6 >$
- $v(3) = \sqrt{37}$
- $\mathbf{v}(3)\cdot\mathbf{v}(3) = v(3)^2 = 37$
- $v(3)^3 = 37^{3/2}$ and $v(3)^{-3} = 37^{-3/2}$
- $\mathbf{T}(3) = 37^{-1/2} < 1, 6 > = \frac{<1,6>}{\sqrt{37}}$
- $\mathbf{a}(3) = < 0, 2 >$
- $\mathbf{v}(3)\cdot\mathbf{a}(3) = 12$
- $\mathbf{T}'(3) = \frac{2}{37^{3/2}} < -6, 1 >$
- $|\mathbf{T}'(3)| = \frac{2}{37}$

If all we ultimately need is the speed of direction change at a particular instant, then all we need do is find $\mathbf{v}(t)$, $\mathbf{a}(t)$, and $v(t)$, then evaluate these at the specified value of t, then compute v^2, $\mathbf{v}\times\mathbf{a}$, and $|\mathbf{v}\times\mathbf{a}|$, and finally divide $|\mathbf{v}\times\mathbf{a}|$ by v^2. Under these circumstances, we *don't* need to find $\mathbf{T}(t)$ or $\mathbf{T}'(t)$ at all. For instance, in the preceding example, suppose our goal had been to find $|\mathbf{T}'(3)|$. As soon as we had $\mathbf{v}(t) = < 1, 2t >$, $\mathbf{a}(t) = < 0, 2 >$, and $v(t) = \sqrt{1+4t^2}$, we could evaluate $\mathbf{v}(3) = < 1, 6 >$, $\mathbf{a}(3) = < 0, 2 >$, and $v(3) = \sqrt{37}$, then compute $v(3)^2 = 37$, $\mathbf{v}(3)\times\mathbf{a}(3) = < 1, 6, 0 > \times < 0, 2, 0 > = < 0, 0, 2 >$, whose magnitude is 2, and then divide: $2 \div 37 = \frac{2}{37}$.

Suppose a particle is moving along a helix centered at the z axis, with position function $\mathbf{r}(t) = < \cos t, \sin t, t^3 >$, and say we want to find the speed of direction change when $t = 5$. We can proceed as follows:

- $\mathbf{v}(t) = < -\sin t, \cos t, 3t^2 >$
- $\mathbf{a}(t) = < -\cos t, -\sin t, 6t >$
- $v(t) = \sqrt{\sin^2 t + \cos^2 t + 9t^4} = \sqrt{1+9t^4}$

- $\mathbf{v}(5) = < -\sin 5, \cos 5, 75 >$
- $\mathbf{a}(5) = < -\cos 5, -\sin 5, 30 >$
- $v(5) = \sqrt{5,626}$
- $v(5)^2 = 5,626$
- $\mathbf{v}(5) \times \mathbf{a}(5) = < 30\cos 5 + 75\sin 5,\ 30\sin 5 - 75\cos 5,\ 1 >$
- $|\mathbf{v}(5) \times \mathbf{a}(5)| = \sqrt{6,526}$
- $|\mathbf{T}'(5)| = \frac{\sqrt{6,526}}{5,626} \approx 0.014359.$

Here, we found $|\mathbf{T}'(5)|$ using Theorem 3, i.e., using the formula $|\mathbf{T}'(t)| = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{v(t)^2}$. If we had used the formula $|\mathbf{T}'(t)| = \frac{|\mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)|}{v(t)^3}$, we would have gotten $|\mathbf{T}'(5)| = \frac{\sqrt{36,715,276}}{5,626^{3/2}}$, which is equivalent, but it would have been more work.

In the above problem, since we were only looking for $|\mathbf{T}'(5)|$, we did not need to find $\mathbf{T}(t)$ and $\mathbf{T}'(t)$. If we had found them, we would have gotten $\mathbf{T}(t) = \frac{<-\sin t, \cos t, 3t^2>}{\sqrt{1+9t^4}}$ and $\mathbf{T}'(t) = \frac{(1+9t^4)\mathbf{a} - 18t^3\mathbf{v}}{(1+9t^4)^{3/2}}$. This is a situation where we would not try to simplify $\mathbf{T}'(t)$, i.e., we would leave it in terms of t, $\mathbf{v}$, and $\mathbf{a}$. Furthermore, in this problem we did not bother to find $|\mathbf{T}'(t)|$ in terms of t. If we had, we would have gotten $|\mathbf{T}'(t)| = \frac{\sqrt{1+36t^2+9t^4}}{1+9t^4}$. But this would have required considerable work.

5. **The Derivative of Speed of Motion**:

So far, we have examined the following derivatives:

- Velocity is the derivative of position. $\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \mathbf{v}(t)$
- Acceleration is the derivative of velocity. $\frac{d}{dt}\mathbf{v}(t) = \mathbf{v}'(t) = \mathbf{a}(t)$
- $\mathbf{T}'(t)$ is the derivative of $\mathbf{T}(t)$. $\frac{d}{dt}\mathbf{T}(t) = \mathbf{T}'(t)$

Now we will discuss one more derivative: The **derivative of speed of motion**, $v(t)$. $\frac{d}{dt}v(t) = v'(t)$.

$v'(t)$ is a scalar-valued function. It is obviously not the same thing as accleration, since acceleration is a vector. You might guess that $v'(t)$ is the magnitude of accleration, but this guess is incorrect. For instance, in the case of the helix $\mathbf{r}(t) = < \cos t, \sin t, t^3 >$, we have $v(t) = \sqrt{1+9t^4}$ and $\mathbf{a}(t) = < -\cos t, -\sin t, 6t >$, so $v'(t) = \frac{1}{2}(1+9t^4)^{-1/2}(36t^3) = \frac{18t^3}{\sqrt{1+9t^4}}$, whereas $a(t) = \sqrt{1+36t^2}$.

Bear in mind, we have discussed two kinds of speed: speed of motion, which is $v(t)$, and speed of direction change, which is $|\mathbf{T}'(t)|$. When we use the word "speed" without specifying which kind we mean, it is always assumed we mean speed of motion. Technically, $v'(t)$ is the *rate of change of speed of motion*, but we can say, more briefly,

$v'(t)$ is the **rate of change of speed**. It tells us how quickly speed (of motion) is changing.

We will *not* address the rate of change of speed of direction change, which would be $\frac{d}{dt}|\mathbf{T}'(t)|$.

We assume, in the following discussion, that we are dealing with smooth motion, so $v(t)$ is nonzero.

Theorem 4: $v'(t) = v(t)^{-1}\ \mathbf{v}(t) \cdot \mathbf{a}(t) = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v(t)}$

Proof (in the case of three dimensions):
Since $v(t) = (x'(t)^2 + y'(t)^2 + z'(t)^2)^{1/2}$,
$v'(t) = \frac{1}{2}(x'(t)^2 + y'(t)^2 + z'(t)^2)^{-1/2}(2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t)) =$
$(x'(t)^2 + y'(t)^2 + z'(t)^2)^{-1/2}(x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)) =$
$v(t)^{-1}\ \mathbf{v}(t) \cdot \mathbf{a}(t) = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v(t)}$.
QED.

Let us confirm Theorem 4 in the case of our helix. We have already established that $\mathbf{v}(t) = < -\sin t, \cos t, 3t^2 >$, $\mathbf{a}(t) = < -\cos t, -\sin t, 6t >$, $v(t) = \sqrt{1+9t^4}$ and $v'(t) = \frac{18t^3}{\sqrt{1+9t^4}}$.
$\mathbf{v}(t) \cdot \mathbf{a}(t) = \cos t \sin t - \cos t \sin t + 18t^3 = 18t^3$, so the theorem is confirmed.

Corollary to Theorem 4: $\mathbf{v}(t) \cdot \mathbf{a}(t) = v(t)v'(t)$

Theorem 5: $\mathbf{T}'(t) = v(t)^{-2}[v(t)\mathbf{a}(t) - v'(t)\mathbf{v}(t)] = \frac{v(t)\mathbf{a}(t) - v'(t)\mathbf{v}(t)}{v(t)^2}$.

Proof:

Since $\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t)$, to differentiate $\mathbf{T}(t)$, we apply the Product Rule, giving us
$\mathbf{T}'(t) = \left(\frac{1}{v(t)}\right)'\mathbf{v}(t) + \frac{1}{v(t)}\mathbf{v}'(t) = \left(\frac{1}{v(t)}\right)'\mathbf{v}(t) + \frac{1}{v(t)}\mathbf{a}(t)$.

$\left(\frac{1}{v(t)}\right)' = \frac{d}{dt}v(t)^{-1} = -v(t)^{-2}v'(t)$

So $\mathbf{T}'(t) = -v(t)^{-2}v'(t)\mathbf{v}(t) + v(t)^{-1}\mathbf{a}(t) = v(t)^{-1}\mathbf{a}(t) - v(t)^{-2}v'(t)\mathbf{v}(t) =$
$v(t)^{-2}[v(t)\mathbf{a}(t) - v'(t)\mathbf{v}(t)] = \frac{v(t)\mathbf{a}(t) - v'(t)\mathbf{v}(t)}{v(t)^2}$.

QED.

Theorem 5 could also be proved using Theorem 1 and the Corollary to Theorem 4:
$\mathbf{T}'(t) = \frac{v(t)^2\mathbf{a}(t) - \mathbf{v}(t)\cdot\mathbf{a}(t)\ \mathbf{v}(t)}{v(t)^3} = \frac{v(t)^2\mathbf{a}(t) - v(t)v'(t)\ \mathbf{v}(t)}{v(t)^3} = \frac{v(t)[v(t)\mathbf{a}(t) - v'(t)\ \mathbf{v}(t)]}{v(t)^3} = \frac{v(t)\mathbf{a}(t) - v'(t)\mathbf{v}(t)}{v(t)^2}$.

Theorem 6: $\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$.

Proof:
Since $\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t)$, $\mathbf{v}(t) = v(t)\mathbf{T}(t)$.
$\mathbf{a}(t) = \frac{d}{dt}\mathbf{v}(t) = \frac{d}{dt}[v(t)\mathbf{T}(t)] = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$, by the Product Rule.
QED.

Earlier, we postponed the proof of Theorem 2. We did so because the proof involves $v'(t)$, which had not yet been discussed. We are now in a position to examine the proof. We will make use of Theorem 6. This is permissible, because Theorem 6 is free-standing–it does not depend on Theorem 2 or any of our other theorems. (If Theorem 6 depended on Theorem 2, we could not use it in proving Theorem 2, since doing so would be circular reasoning, which is invalid.)

Proof of Theorem 2:

Since $\mathbf{v}(t) = v(t)\mathbf{T}(t)$ and $\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$,
$\mathbf{v}(t) \times \mathbf{a}(t) = [v(t)\mathbf{T}(t)] \times [v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)]$.

By the Scalar Multiple Rule for cross products (discussed in Section 12.4), we can factor out $v(t)$, giving us $v(t)\{\mathbf{T}(t) \times [v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)]\}$.

By the Distributive Property, $\mathbf{T}(t) \times [v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)] =$
$[\mathbf{T}(t) \times v'(t)\mathbf{T}(t)] + [\mathbf{T}(t) \times v(t)\mathbf{T}'(t)]$.

By the Scalar Multiple Rule, we get
$v'(t)[\mathbf{T}(t) \times \mathbf{T}(t)] + v(t)[\mathbf{T}(t) \times \mathbf{T}'(t)]$.

$\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ by Theorem 2 of Section 12.4.

So now we have $v'(t)[\mathbf{0}] + v(t)[\mathbf{T}(t) \times \mathbf{T}'(t)] =$
$\mathbf{0} + v(t)[\mathbf{T}(t) \times \mathbf{T}'(t)] = v(t)[\mathbf{T}(t) \times \mathbf{T}'(t)]$.

Finally, we have $v(t)\{v(t)[\mathbf{T}(t) \times \mathbf{T}'(t)]\} = v(t)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$

QED.