5.5 The Substitution Rule for Integration

The Substitution Rule: If $u = g(x)$ is a differentiable function whose range is on interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Procedure:

1. Given an indefinite integral involving a composite function $f(g(x))$, identify an inner function $u = g(x)$ such that a constant multiple of $g'(x)$ appears in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)$ in the integral.
3. Evaluate the new indefinite integral with respect to u.
4. Write the result in terms of x using **u = g(x).** (In other words, back substitute.)

Example: Find the integrals:

a) $\int (cos^3 x \sin x)dx$

let **u** = cos(x)

then **du** = −sin(x)dx → **−du** = dx

then substitute u^3 for $cos^3(x)$ and $-du$ for dx

$\int u^3 \cdot -du = -\int u^3 du = -\frac{u^4}{4} + C$

Now back substitute.

$\int (cos^3 x \sin x)dx = -\frac{cos^4(x)}{4} + C$

b) $\int 10e^{10x}dx \rightarrow \int e^{10x} \cdot 10dx$

let **u** = 10^x

then **du** = 10dx

then substitute e^u for e^{10x} and **du** for 10dx

$\int e^u du = e^u + C$

Now back substitute.

$\int 10e^{10x}dx = e^{10x} + \mathrm{C}$

Sometimes the choice for a **u-substitution** is not so obvious. The next example illustrates this.

Example: Find $\int \frac{x}{\sqrt{x+1}}dx$

Let **u** = x+1 ⇒ **u – 1** = x

du = dx.

$$\int \frac{x}{\sqrt{x=1}}dx = \int \frac{u-1}{\sqrt{u}}du = \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du = \int \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du = \frac{2u^{\frac{3}{2}}}{3} - 2u^{\frac{1}{2}} + C$$

Now back substitute, u = x + 1 ⇒ $\frac{2(x+1)^2}{3} - 2(x+1)^{\frac{1}{2}} + \mathrm{C}$

Example: Find $\int \sqrt{1+x^2}\cdot x^5 dx$ Write x^5 as $x^4 \cdot x$ and let $\mathbf{u} = 1 + x^2$

$\mathbf{du} = 2xdx$ and $\frac{1}{2}\boldsymbol{du} = xdx$

Also notice that $\boldsymbol{x^2} = \boldsymbol{u-1}$ which can also be written as $(\boldsymbol{x^2})^2 = (\boldsymbol{u-1})^2 \Rightarrow \boldsymbol{x^4} = (\boldsymbol{u-1})^2$

Therefore, $\int \sqrt{1+x^2}\cdot x^5 dx = \int \sqrt{1+x^2}\cdot x^4 \cdot xdx$

$= \int \sqrt{u}\cdot (u-1)^2 \cdot \frac{1}{2} du$ Move the constant ½ to the outside and expand the binomial.

$= \frac{1}{2}\int u^{\frac{1}{2}}\cdot (u^2 - 2u + 1)du = \frac{1}{2}\int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$

$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + C$ *or* $\frac{1}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}} + \frac{1}{3}u^{\frac{3}{2}} + C$ (back substitute)

$= \boldsymbol{\frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C}$

When evaluating a **definite** integral by substitution, back substitution is not necessary. We change the limits of integration when the variable is changed.

The Substitution Rule for Definite Integrals: If ***g'(x)*** is continuous on [a, b] and ***f*** is continuous on the range of **u = g(x)**, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example: Evaluate the following integrals.

a) $\int_0^4 \frac{x}{x^2+1}dx$

let $\mathbf{u} = x^2 + 1$ then $\mathbf{du} = 2xdx \Rightarrow \frac{1}{2}du = xdx$

Now change the limits of integration:

$x = 0 \Rightarrow u = 1$

$x = 4 \Rightarrow u = 17$

This gives us:

$\int_0^4 \frac{x}{x^2+1}dx = \int_1^{17} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int_1^{17}\frac{du}{u} =$

$\frac{1}{2}(\ln u)\Big]_1^{17} = \frac{1}{2}(\ln(17) - \ln(1)) =$

$\boldsymbol{\frac{1}{2}\ln(17)}$

b) $\int_0^{\frac{\pi}{2}}(sin^4(x)\cos(x))dx$

let $\mathbf{u} = \sin(x)$ then $\mathbf{du} = \cos(x)dx$

Now change the limits of integration:

$x = 0 \Rightarrow u = \sin(0) = 0$

$x = \frac{\pi}{2} \Rightarrow u = \sin\left(\frac{\pi}{2}\right) = 1$

This gives us:

$\int_0^{\frac{\pi}{2}} sin^4(x)\cos(x)\,dx = \int_0^1 u^4 du =$

$\frac{u^5}{5}\Big]_0^1 =$

$\boldsymbol{\frac{1}{5}}$