2.5 Continuity

In this section we will discuss the definition of continuity and introduce one of the most important theorems of Mathematics: The Intermediate Value Theorem (IVT).

Functions that have a limit found by calculating the value of the function at ***a*** are said to be continuous at ***a***.

Definition: A function ***f*** is continuous at a number ***a*** if

$$\lim_{x\to a} f(x) = f(a)$$

Notice that this definition requires three things for ***f*** to be continuous ***a***:

1. f(a) is defined (that means f(a) exists)
2. $\lim_{x\to a} f(x)\ exists$
3. $\lim_{x\to a} f(x) = f(a)$

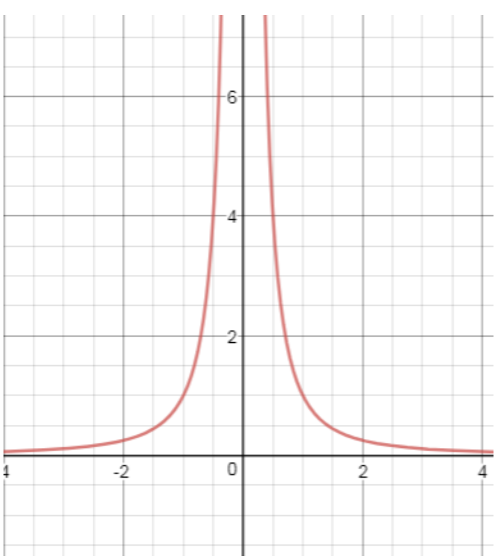
Example: $f(x) = \frac{x^2-9}{x-3}$ f(x) is not continuous at x = 3 because 3 is not in the domain of ***f***. Also notice that $f(3) = undefined$

If ***f*** is defined near ***a***, we say that ***f*** is discontinuous at ***a*** if ***f*** is not continuous at ***a***.

Geometrically, you can think of a function that is continuous at every number in an interval as function whose graph has no break in it: the the graph can be drawn without removing your pen/pencil from the paper.

Example: Where is the function discontinuous? $f(x) = \begin{Bmatrix} \frac{1}{x^2} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{Bmatrix}$

Graph this function: The function is discontinuous at x = 0 because $\lim_{x\to 0} f(x) \neq f(0)$.

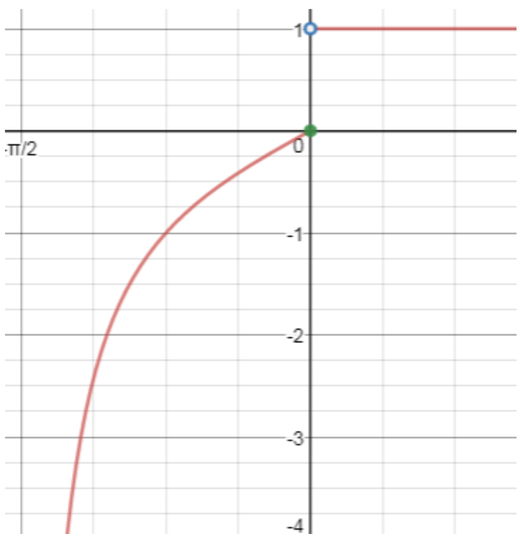


Definition: A function ***f*** is continuous from the right at a number ***a*** if $\lim_{x\to a^+} f(x) = f(a)$

And f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$

Example: $f(x) = \begin{cases} \tan(x) & if\ \ x \le 0 \\ 1 & if\ x\ > 0 \end{cases}$

Graph this function:



This function is continuous from the left at $x = 0$ because $\lim_{x \to 0^-} f(x) = f(0)$

But it is discontinuous from the right because $\lim_{x \to 0^+} f(x) \neq f(0)$

Definition: A function f is continuous on an interval if it is continuous at every number in the interval.

Theorem: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. $c \cdot f$
4. $f \cdot g$
5. $\frac{f}{g}$ if $g(a) \neq 0$

Theorem: a) All polynomials are continuous everywhere; that is, they are continuous on $(-\infty, \infty)$.
b) A rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Theorem: The following types of functions are continuous at every number in their domain.

- Polynomials
- Rational functions
- Root functions
- Exponential functions
- Logarithmic functions
- Trigonometric functions
- Inverse Trigonometric functions

Example: Where is the function $f(t) = \frac{e^t+\sin(t)}{2+\cos(\pi t)}$ continuous? First let's analyze each part of this function for continuity, then make a conclusion.

- $\sin(t)$ *is continuous everywhere* $(-\infty, \infty)$ *because it is a trigonometric function*
- e^t *is continuous everywhere* $(-\infty, \infty)$ *because it is an exponential function.*
- *notice that* 2 *and* $\cos(\pi t)$ *are continuous everywhere therfore their sum is also continuous*
- *since* $\cos(\pi t) \geq -1$ *for all t in it's domain, then* $2 + \cos(\pi t) > 0$ *for all t*
- *thus,* $f(t)$ *is continuous everywhere on its domain.*

Theorem: If ***g*** is continuous at ***a*** and ***f*** is continuous at ***g(a)***, then the composite function ***f o g*** given by ***(f o g) = f(g(x))*** is continuous at ***a.***

Example: Where is the function $f(t) = \arcsin(1 + 2t)$ continuous?

Let $h(t) = \arcsin(t)$ and $g(t) = 1 + 2t$ then $f(t) = h(g(t))$. We know that $g(t)$ is continuous everywhere because it is a polynomial. We also know that $h(t)$ is continuous on it domain which is [1, -1]. So as long as -1 ≤ $g(t)$ ≤ 1, then $f(t)$ is continuous.

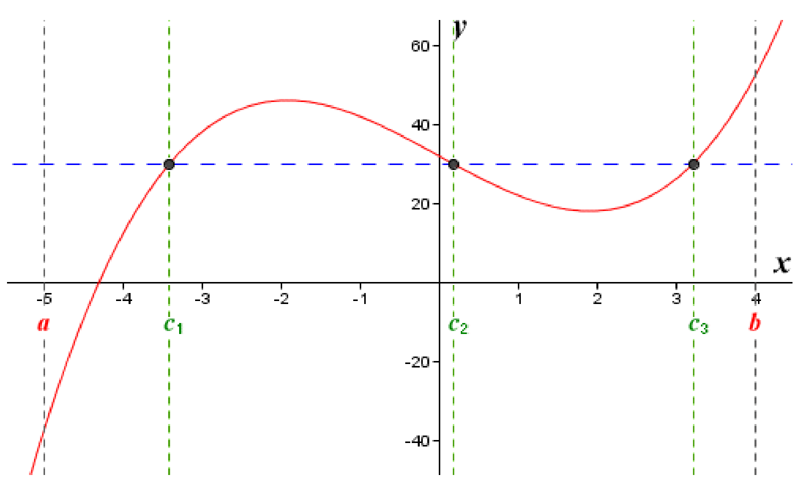
$$-1 \leq 1 + 2t \leq 1 \qquad \text{solve for t}$$
$$-2 \leq 2t \leq 0$$
$$-1 \leq t \leq 0$$

Therefore, $f(t)$ is continuous on [-1, 0].

The Intermediate Value Theorem: Suppose ***f*** is continuous on the closed interval [a, b] and let N be any number between f(a) and f(b), where f(a) ≠ f(b). Then there exists a number, ***c***, such that ***f(c)*** = N.

This theorem says that the graph must cross a horizontal line ***y = N***, where f(a) < N< f(b), in the closed interval [a, b] if ***f*** is continuous. Notice that the function must be continuous for the IVT to work. Also the function can cross ***y = N*** more than once.

Graphically:



The graph of the function $f(x) = x^3 - 11x + 32$

Example: Use the IVT to show that there is a root of the function in the specified interval.

$\mathbf{sin}(\boldsymbol{x}) = \boldsymbol{x}^2 - \boldsymbol{x}$, $(\mathbf{1}, \mathbf{2})$, $\textit{Notice that } \sin(x), x^2, \textit{and } x \textit{ are continuous } \therefore \mathbf{sin}(\boldsymbol{x}) = \boldsymbol{x}^2 - \boldsymbol{x}$ is continuous.

$\mathbf{sin}(\boldsymbol{x}) = \boldsymbol{x}^2 - \boldsymbol{x}$

$\mathbf{sin}(\boldsymbol{x}) - \boldsymbol{x}^2 + \boldsymbol{x} = \mathbf{0}.$ So we need to find a number, ***c***, between 1 and 2 such that ***f(c)=0***

If we let a = 1, b = 2 and N = 0, then we have the following:

$f(1) = \sin(1) - 1^2 + 1 \approx .84147 - 1 + 1 \approx .84147 > 0$

$f(2) = \sin(2) - 2^2 + 2 \approx .904297 - 4 + 2 \approx -1.0907 < 0$

Thus f(2) < 0 < f(1); which means that N = 0 is a number between f(1) and f(2). Since $\mathbf{sin}(\boldsymbol{x}) = \boldsymbol{x}^2 - \boldsymbol{x}$ is continuous, the IVT says that there is a number ***c*** between x = 1 and x = 2 such that ***f(c) = 0.*** Which means that the equations $\mathbf{sin}(\boldsymbol{x}) = \boldsymbol{x}^2 - \boldsymbol{x} = \mathbf{0}$ has at least one root (solution or zero) in the interval (1, 2).