7.7 Approximate Integration

Sometimes it is difficult or impossible to find the exact value of a definite integral. For this reason we introduce methods to find the approximate solution.

In earlier sections we introduced the idea using rectangles with height given by the right endpoint, the left endpoint, and the midpoint of each interval. In this section we will learn the **trapezoidal rule.**

The trapezoidal rule results from averaging the approximations given by the left endpoint of an interval and the right endpoint of an interval.

If the i^{th} subinterval is given by $[x_{i-1}, x_i]$, then the approximation using the left endpoint is:

$$L_n = \int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

and the approximation using the right endpoint is:

$$R_n = \int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

The Trapezoidal Rule says:

$$\int_a^b f(x)dx \approx \frac{1}{2}[L_n + R_n] = \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right]$$

$$= \frac{1}{2}\Delta x\left[\sum_{i=1}^{n} f(x_{i-1}) + \sum_{i=1}^{n} f(x_i)\right] = \frac{1}{2}\Delta x\left[\sum_{i=1}^{n} f(x_{i-1}) + f(x_i)\right]$$

$$= \frac{1}{2}\Delta x[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + (f(x_2) + f(x_3)) + \cdots (f(x_{n-1}) + f(x_n))]$$

Trapezoidal Rule:

$$\int_a^b \boldsymbol{f(x)dx} \approx \boldsymbol{T_n} = \frac{\boldsymbol{\Delta x}}{\boldsymbol{2}}[\boldsymbol{f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)}]$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

<u>Comparing the Midpoint Rule and the Trapezoidal Rule</u>:

1. The Trapezoidal and Midpoint Rules are much more accurate than the left and right endpoint approximations.
2. The size of error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

Error Bounds:

Suppose $|f''(x)| \leq k,\ for\ a \leq x \leq b.$ If $E_T\ and\ E_M$ are the errors in the Trapezoidal and Midpoint Rules respectively, then

$$|E_T| \le \frac{k(b-a)^3}{12n^2} \quad and \quad |E_M| \le \frac{k(b-a)^3}{24n^2}$$

Example: Use the Trapezoidal Rule to approximate the given integral with the specified value of n. (Round to **six** decimal places).

a) $\int_1^3 e^{\frac{1}{x}}dx,\ \ n = 8$ b) $\int_0^1 \sqrt{x + x^3}dx,\ \ n = 10$

(a) Solution

$\Delta x = \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}$ $x_0 = 1 + 0 \cdot \frac{1}{4} = 1,\ \ x_1 = 1 + 1 \cdot \frac{1}{4} = 1.25,\ \ x_2 = 1.5,\ \ x_3 = 1.75,\ \ x_4 = 2,\ \ x_5 = 2.25,$
$x_6 = 2.5,\ \ x_7 = 2.75,\ \ and\ x_8 = 3$

$$\int_1^3 e^{\frac{1}{x}}dx \approx \frac{\frac{1}{4}}{2}[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + 2f(2) + 2f(2.25) + 2f(2.5) + 2f(2.75) + f(3)]$$

$$\approx \frac{1}{8}\left[e^{\frac{1}{1}} + 2e^{\frac{1}{1.25}} + 2e^{\frac{1}{1.5}} + 2e^{\frac{1}{1.75}} + 2e^{\frac{1}{2}} + 2e^{\frac{1}{2.25}} + 2e^{\frac{1}{2.5}} + 2e^{\frac{1}{2.75}} + e^{\frac{1}{3}}\right]$$

$$\approx \frac{1}{8}[2.718282 + 4.451082 + 3.895468 + 3.541590 + 3.297443 + 3.119247 + 2.983649 + 2.877102 + 1.395612] \approx \mathbf{3.534934}$$

(b) Solution

$$\Delta x = \frac{1}{10}\ or\ .1 \quad x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, x_4 = 0.4, x_5 = 0.5, x_6 = 0.6, x_7 = 0.7, x_8 = 0.8, x_9 = 0.9, x_{10} = 1.0$$

$$\int_0^1 \sqrt{x + x^3}dx \approx \frac{1}{20}[f(0) + 2f(.1) + 2f(.2) + 2f(.3) + 2f(.4) + 2f(.5) + 2f(.6) + 2f(.7) + 2f(.8) + 2f(.9) + f(1)]$$

$$\approx \frac{1}{20}\left[\sqrt{0 + 0^3} + 2\sqrt{.1 + .1^3} + 2\sqrt{.2 + .2^3} + \cdots + 2\sqrt{.9 + .9^3} + \sqrt{1 + 1^3}\right]$$

$$\approx \frac{1}{20}[0 + .635610 + . + .912140 + 1.143678 + 1.362351 + 1.581139 + 1.806654 + 2.042547 + 2.290851 + 2.552646 + 1.414214] \approx \mathbf{0.787092}$$

Example: How large should we take **n** in order to guarantee that the trapezoidal rule is accurate to 0.0001 of the following integral?

$$\int_{0.5}^1 \sqrt{x + x^3}\,dx$$

Since we are looking for accuracy we need to use the Error Bound formula for the Trapezoidal Rule.

$|E_T| \le \frac{k(b-a)^3}{12n^2}$ First find f''. $f(x) = (x + x^3)^{\frac{1}{2}}$ $f'(x) = \frac{1}{2}(x + x^3)^{-\frac{1}{2}}(1 + 3x^2) = \frac{1+3x^2}{2\sqrt{x+x^3}}$

$$f''(x) = \frac{2\sqrt{x + x^3}(6x) - (1 + 3x^2)((x + x^3)^{-\frac{1}{2}}(1 + 3x^2)}{\left(2\sqrt{x + x^3}\right)^2} = \cdots = \frac{3x^4 + 6x^2 - 1}{4(x + x^3)^{\frac{3}{2}}}$$

Plot the graph of f'' and you will see that $|f''(x)| \leq 0.8 \;\; for \; 0.5 \leq x \leq 1$.

Then

$$|E_T| \leq \frac{k(b-a)^3}{12n^2} \quad \Rightarrow \quad \frac{.8(1-0.5)^3}{12n^2} < 0.0001 \quad \Rightarrow \quad \frac{.8(.125)}{12(0.0001)} < n^2$$

$n^2 > \frac{.8(.125)}{12(0.0001)} \quad \Rightarrow \quad n^2 > 83.\overline{3} \quad \Rightarrow \quad \boldsymbol{n > 9.1287}$ … therefore **n = 10** will yield the desired accuracy.