2.1 The Tangent and Velocity Problem

In this section we see how limits arise when we attempt to find the tangent to a curve or a velocity of an object.

Notice: A tangent to a curve is a line that touches the curve. This tangent line can be different for different points on the curve. Also notice that a tangent line should have the same direction as the curve at the point of contact.

In the following example we will find the tangent line and find the distinction between a tangent line and a secant line.

The Tangent Line Problem

Example: Find the equation of the tangent line to the parabola $y = x^2$ at the point (3, 9). Also make a table that approximates $x = 3$ from the left and right side.

Find the slope of $y = x^2$ at (3, 9). Use the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$. Well … we only have one point an we need two points for the slope formula. To fix this, we compute an approximate slope (m) by choosing a nearby point to point (3, 9) that is also on the parabola $y = x^2$. Let point A be (3, 9) and point B be (3.5, 12.25). Now find the slope of AB.

$$m_{AB} = \frac{12.25 - 9}{3.5 - 3} = \frac{3.5}{.5} = 6.5$$

Now, let's pick a point closer to 3 than 3.5. Let's use point B as (3.25, 10.5625). Now find the slope of AB.

$$m_{AB} = \frac{10.5625 - 9}{3.25 - 3} = \frac{1.5625}{.25} = 6.25$$

If we continue this process several times, approaching $x = 3$ from the right, we get the following table

x	m_{AB}
3.5	6.5
3.25	6.25
3.1	6.1
3.01	6.01
3.001	6.001

If we use this process several times, approaching $x = 3$ from the left. We get the following table

x	m_{AB}
2.5	5.5
2.75	5.75
2.9	5.9
2.99	5.99
2.999	5.999

From the tables above it appears that the closer we get to $x = 3$, the slope m_{AB} gets closer to 6. This means that the slope of the line tangent to $y = x^2$ at $x = 3$ is 6. We can also say that the slope of the tangent line is the limit of the slopes of the secant lines, which can be expressed in the following notation:

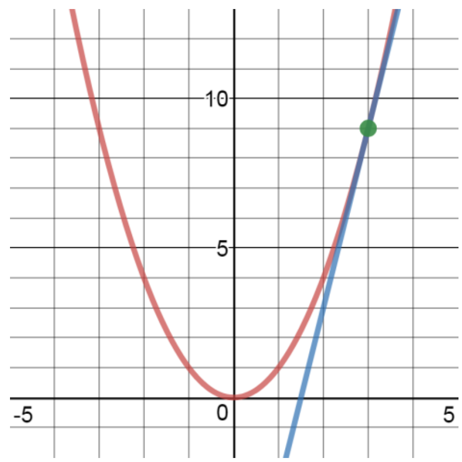
$$\lim_{A \to B} m_{AB} = m \quad and \quad \lim_{x \to 3} \frac{x^2 - 3}{x - 3} = 6$$

Since the tables above suggest that the slope of the tangent line is equal to 6, we can use the point-slope form of the equation of a line $\boldsymbol{y - y_1 = m(x - x_1)}$ to write the equation of the tangent line passing through (3, 9) as $y - 9 = 6(x - 3)$

$$y - 9 = 6x - 18$$

$$\boxed{y = 6x - 9}$$

This line, $y = 6x - 9$, is the line tangent to the curve $y = x^2$ at the point (3, 9). Below is the graph of the tangent line, the curve and the point of tangency.



Notice: If you plot all of the secant lines as x approaches 3, they should get closer and closer to the tangent line $y = 6x - 9$.

The Velocity Problem

In this part of the section we discuss how to define "instantaneous" velocity. We will attempt to describe instantaneous velocity by starting with "average" velocity.

$$\boldsymbol{average\ velocity} = \frac{\boldsymbol{change\ in\ position}}{\boldsymbol{time\ elapsed}}$$

Notice: The average velocity has two different times so we have to think of the average velocity as the change in velocity between two different points in time.

Example: If the position of a ball (that has been projected) is given by $\boldsymbol{y = 40t - 16t^2}$ $(where\ t = time\ in\ seconds\ and\ postion\ is\ in\ feet)$, what is the average velocity of the ball between the times $t = 3$ seconds and $t = 4$ seconds? Let the units be feet per second (ft/sec)

The change in position = y(4) – y(3)
$= [40(4) - 16(4)^2] - [40(3) - 16(3)^2]$
= [160 – 256] – [120 – 144]
= -96 – (-24)
= -72 (the ball is falling)

The time elapsed = 4 – 3 seconds
= 1 second

Average Velocity = $\frac{-72\ feet}{1\ second}$ = - 72 ft/sec

Notice that the example above shows the average velocity between t = 3 and t = 4 seconds of the ball is -72 ft/sec. BUT what if we wanted to know the exact velocity, "instantaneous velocity" at time t = 3?

We can choose smaller time intervals and approach t = 3. For instance, instead of finding the average velocity between t = 3 and t = 4, we could find the average velocity between t = 3 and t = 3.001 or between t = 3 and t = 3.0001 – just like we did in finding the slope of the tangent line in the last example.

In other words, the instantaneous velocity when t = 3 is defined to be the limiting value of these average velocities over smaller and smaller periods of time that start at t = 3. The following table show a clearer idea of that is happening.

Time Interval	Average velocity (ft/sec)
$3 \le t \le 3.5$	-64
$3 \le t \le 3.1$	-57.6
$3 \le t \le 3.01$	-56.16
$3 \le t \le 3.001$	-56.016
$3 \le t \le 3.0001$	-56.0016

The table above suggests that the instantaneous velocity at time t = 3 is **-56 ft/sec.**