7.3 Trigonometric Substitution

In the following table we have a list of trigonometric substitutions that are effective for the given radical expressions because of the specified trigonometric identities. In each case the restriction on $\boldsymbol{\theta}$ is imposed to ensure that the function that defines the substitution is 1 – to – 1.

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x=a\sin\theta,\quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta,\quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta,\quad 0\le\theta<\frac{\pi}{2},\ \text{or}\ \pi\le\theta<\frac{3\pi}{2}$	$\sec^2\theta-1=\tan^2\theta$

Below is a table showing how to use the Trigonometric Substitutions. Using these types of substitution is called **inverse substitution**. Try to match the type of radical in your integral with one of the examples below.

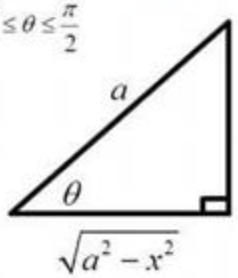
TYPE 1

1. $\sqrt{a^2-x^2}$ Let $x=a\sin\theta$

$$\sqrt{a^2-x^2}=\sqrt{a^2-(a\sin\theta)^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2(1-\sin^2\theta)}=\sqrt{a^2\cos^2\theta}=a\cos\theta$$

$$x=a\sin\theta\Rightarrow\frac{x}{a}=\sin\theta$$

$$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$$



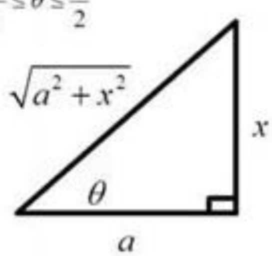
TYPE 2

2. $\sqrt{a^2+x^2}$ Let $x=a\tan\theta$

$$\sqrt{a^2+x^2}=\sqrt{a^2+(a\tan\theta)^2}=\sqrt{a^2+a^2\tan^2\theta}=\sqrt{a^2(1+\tan^2\theta)}=\sqrt{a^2\sec^2\theta}=a\sec\theta$$

$$x=a\tan\theta\Rightarrow\frac{x}{a}=\tan\theta$$

$$\frac{-\pi}{2}\le\theta\le\frac{\pi}{2}$$



TYPE 3

2. $\sqrt{x^2-a^2}$ Let $x=a\sec\theta$

$$\sqrt{x^2-a^2}=\sqrt{(a\sec\theta)^2-a^2}=\sqrt{a^2\sec^2\theta-a^2}=\sqrt{a^2(\sec^2\theta-1)}=\sqrt{a^2\tan^2\theta}$$

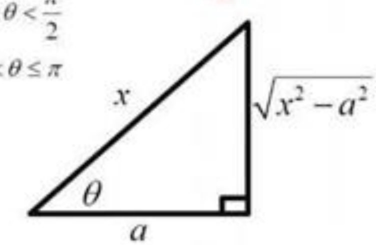
$\sqrt{x^2-a^2}=a\tan\theta$ for $x>a$

$\sqrt{x^2-a^2}=-a\tan\theta$ for $x<-a$

$$x=a\sec\theta\Rightarrow\frac{x}{a}=\sec\theta$$

$$0\le\theta<\frac{\pi}{2}$$

$$\frac{\pi}{2}<\theta\le\pi$$



Example: Evaluate

$$\int\frac{\sqrt{9-x^2}}{x^2}\,dx$$

Notice this problem matches up with type 1. Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta d\theta$

$$\sqrt{9-x^2} = \sqrt{9-9sin^2\theta} = \sqrt{9(1-sin^2\theta)} = \sqrt{9cos^2\theta} = 3|cos\theta| = 3cos\theta$$

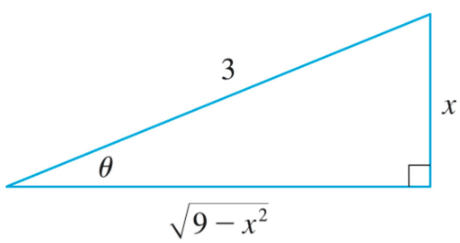
(Note that $\cos\theta \ge 0$ because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.) Thus the inverse substitution rule gives

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{3\cos\theta}{9sin^2\theta} 3\cos\theta d\theta = \int \frac{9cos^2\theta}{9sin^2\theta}d\theta$$

$$= \int cot^2\theta d\theta = \int (csc^2\theta - 1)d\theta$$

$$= -\cot(\theta) - \theta + C$$

Since this is an indefinite integral we must return to the original variable $\boldsymbol{x}$. This can be done by either using trigonometric identities to express $\cot(\theta)$ in terms of $\sin\theta = \frac{x}{3}$ or by drawing a diagram similar to the diagram in TYPE 1 (see below) where θ is interpreted as an angle of a right triangle. Since $\sin\theta = \frac{x}{3}$ we can label the side opposite θ as $\boldsymbol{x}$ and the hypotenuse as 3 which, by using the Pythagorean Theorem, gives us $\sqrt{9-x^2}$ for the adjacent side.



Therefore $\cot\theta = \frac{\sqrt{9-x^2}}{x}$ (Although $\Theta > 0$ in the diagram, this expression for $\cot\Theta$ is valid even when $\Theta < 0$.) Since $\sin\theta = \frac{x}{3}$, we have $\theta = sin^{-1}\left(\frac{x}{3}\right)$ and so by back substituting we get:

$$\int \frac{\sqrt{9-x^2}}{x^2}dx = -\boldsymbol{\frac{\sqrt{9-x^2}}{x}} - \boldsymbol{sin^{-1}\left(\frac{x}{3}\right) + C}$$

Example: Evaluate

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x-2}dx$$

This is not an obvious example. We have to complete the square of $x^2 + 4x - 5$.

$$\int_1^4 \frac{\sqrt{(x-2)^2-9}}{x+2}dx$$

Let **u** = x + 2, then **du** = **dx** and the limits of integration change: when x = 1 then **u = 3**, x = 4 then **u = 6**

$$\int_3^6 \frac{\sqrt{u^2-9}}{u}du$$

This matches up with TYPE 3 and we can now do a secant substitution where **u = 3sec Θ** and **du = 3sec Θ tan Θ dΘ.** Changing the limits of integration again, when u = 3 then **Θ = 0** and when u = 9, $\boldsymbol{\Theta = \frac{\pi}{3}}$, so we have the following:

$$\int_1^4 \frac{\sqrt{(x-2)^2-9}}{x+2}\,dx = \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du = \int_0^{\frac{\pi}{3}} \frac{\sqrt{(3\sec(\theta))^2-9}}{3sec\theta}\cdot 3\sec(\theta)\tan(\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{9sec^2(\theta)-9}}{3sec\theta}\cdot 3\sec(\theta)\tan(\theta)\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\sqrt{9(sec^2(\theta)-1)}}{\cancel{3sec\theta}}\cdot \cancel{3\sec(\theta)}\tan(\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{9tan^2(\theta)}\cdot\tan(\theta)\,d\theta = \int_0^{\frac{\pi}{3}} 3\tan(\theta)\cdot\tan(\theta)\,d\theta = 3\int_0^{\frac{\pi}{3}} tan^2(\theta)d\theta = 3\int_0^{\frac{\pi}{3}} (sec^2(\theta)-1)d\theta$$

$$= 3(\tan(\theta)-\theta)\Big|_0^{\frac{\pi}{3}} = 3\left[\left(\tan\left(\frac{\pi}{3}\right)-\frac{\pi}{3}\right)-(\tan(0)-0)\right] = 3\left[\sqrt{3}-\frac{\pi}{3}\right] = \mathbf{3\sqrt{3}-\pi}$$

Example: Evaluate

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx$$

This is a TYPE 2 trigonometric substitution so we let $\boldsymbol{x = 2tan(\theta)}$ and $\boldsymbol{dx = 2sec^2(\theta)d\theta,}$ therefore

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{2sec^2(\theta)d\theta}{4tan^2(\theta)\sqrt{4tan^2(\theta)+4}} = \int \frac{2sec^2(\theta)d\theta}{4tan^2(\theta)\sqrt{4(tan^2(\theta)+1)}}$$

$$= \int \frac{2sec^2(\theta)d\theta}{4tan^2(\theta)\cdot 2\sqrt{(tan^2(\theta)+1)}} = \int \frac{2sec^2(\theta)d\theta}{8tan^2(\theta)\sqrt{sec^2(\theta)}} = \frac{1}{4}\int \frac{\sec(\theta)}{tan^2(\theta)}d\theta$$

Rewrite $\frac{\sec(\theta)}{tan^2(\theta)}$ in terms of cosine and sine. $\frac{\sec(\theta)}{tan^2(\theta)} = \frac{\frac{1}{\cos(\theta)}}{\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2} = \frac{1}{\cos(\theta)}\cdot\frac{cos^2(\theta)}{sin^2(\theta)} = \frac{\cos(\theta)}{sin^2(\theta)}$ Therefore

$$\frac{1}{4}\int \frac{\sec(\theta)}{tan^2(\theta)}d\theta = \frac{1}{4}\frac{\cos(\theta)}{sin^2(\theta)}d\theta$$

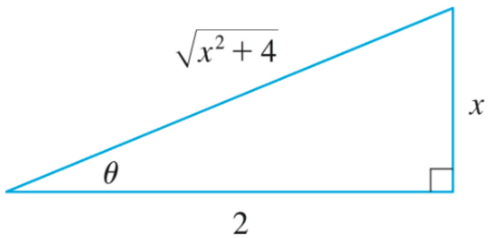
Let $\mathbf{u} = \sin(\theta)$ thus $\mathbf{du} = \cos(\theta)\,d\theta$ using **u – substitution** we get:

$$= \frac{1}{4}\int \frac{1}{u^2}du = \frac{1}{4}\left[-\frac{1}{u}+C\right] = -\frac{1}{4u}+C$$

Remember $\mathbf{u} = \sin(\theta)$ so we back substitute and we get:

$$= -\frac{1}{4\sin(\theta)}+C = -\frac{1}{4}\csc(\theta)+C$$

Now we must write $\csc(\theta)$ in terms of $\mathbf{x}$ so we use the diagram below to get that $\csc(\theta) = \frac{\sqrt{x^2+4}}{x}$



Therefore:

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = -\frac{\sqrt{\boldsymbol{x^2+4}}}{\boldsymbol{x}}+\boldsymbol{C}$$