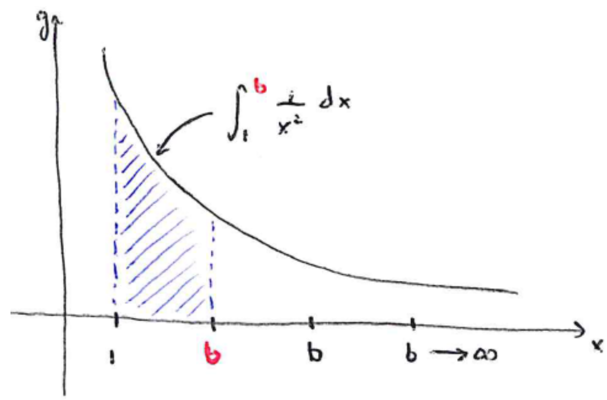
7.8 Improper Integrals

In this section we will deal with the idea of definite integrals where the interval is infinite and also where the function ***f*** has an infinite discontinuity in [a, b]. In each case the integral is called an improper integral.

Infinite Integrals:

Consider the integral:

$$\int_1^b \frac{1}{x^2}\,dx \quad for\ b > 1$$



Notice that this integral give the area of the region bounded by the curves $y = \frac{1}{x^2}$ and the x – axis between x = 1 and x = b. Thus

$$\int_1^b \frac{1}{x^2}\,dx = \int_1^b x^{-2}\,dx = -\frac{1}{x}\bigg|_1^b = \mathbf{1} - \frac{\mathbf{1}}{\boldsymbol{b}}$$

If we increase b, the area under the curve also increases. But what happens to the area as b becomes arbitrarily large? In other words, $b \to \infty$.

$$\lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = \mathbf{1}$$

This says that a curve of infinite length that bounds a region has a finite area! Therefore we can say,

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

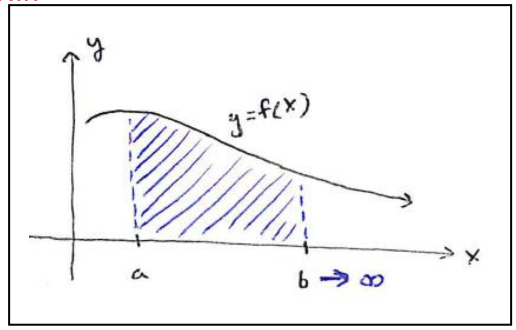
This is an improper integral because ∞ appears in the upper limit.

To evaluate $\int_a^\infty f(x)dx$, we first integrate over a finite interval [a, b] and then take the limit as $b \to \infty$.

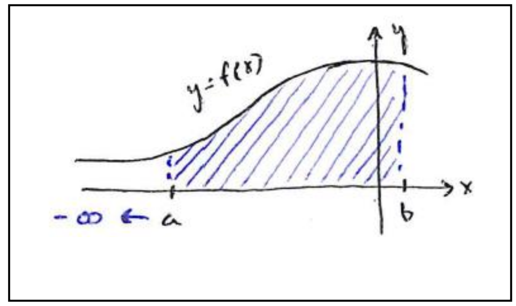
Similar procedures are used to evaluate $\int_{-\infty}^b f(x)dx\ \ and\ \ \int_{-\infty}^\infty f(x)dx$.

Definition of an Improper Integral of Type 1:

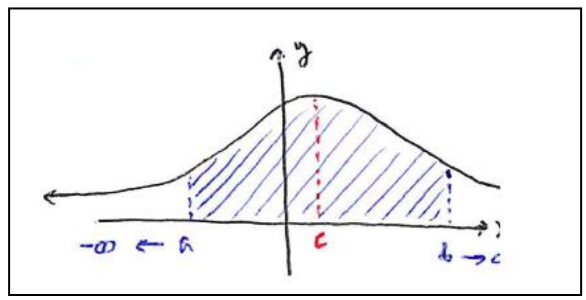
1. If f is continuous on [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$



2. If f is continuous on (- ∞, b], then $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} f(x)dx$



3. If f is continuous on (- ∞, ∞), then
   $\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \int_a^c f(x)dx + \lim_{b\to\infty} \int_c^b f(x)dx$
   where **c** is any real number.



If the **limit** in cases 1 – 3 **exist**, then the improper integral **converges**: otherwise they **diverge**.

Example: Evaluate each integral.

a) $\int_0^\infty e^{-3x}\,dx$

a) $\int_0^\infty e^{-3x}dx = \lim_{b\to\infty} \int_0^b e^{-3x}dx = \lim_{b\to\infty} \left[-\frac{1}{3}e^{-3x}\right]_0^b = \lim_{b\to\infty} -\frac{1}{3}(e^{-3b} - e^0)$

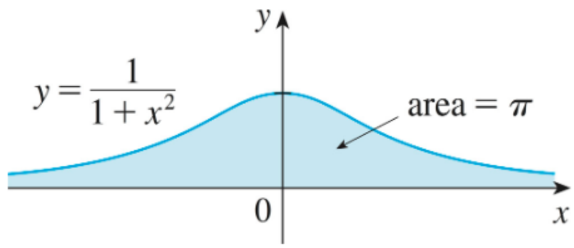
$= \lim_{b\to\infty} \frac{1}{3}(1 - e^{-3b}) = \frac{1}{3}\left(1 - \lim_{b\to\infty} e^{-3b}\right) = \frac{1}{3}(1-0) = \mathbf{\frac{1}{3}}$

b) $\int_{-\infty}^\infty \frac{1}{1+x^2}$

b) $\int_{-\infty}^\infty \frac{1}{1+x^2}dx = \lim_{a\to-\infty} \int_a^c \frac{1}{1+x^2}dx + \lim_{b\to\infty} \int_c^b \frac{1}{1+x^2}dx$ (since **c** can be any real number, we choose **c = 0.**)

$= \lim_{a\to-\infty} \tan^{-1} x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1} x\Big|_0^b = \lim_{a\to\infty}[\tan^{-1} 0 - \tan^{-1} a] + \lim_{b\to\infty}[\tan^{-1} 0 - \tan^{-1} b]$

$= \lim_{a\to\infty}[0 - \tan^{-1} a] + \lim_{b\to\infty}[0 - \tan^{-1} b] = \lim_{a\to\infty}[\tan^{-1} a] + \lim_{b\to\infty}[\tan^{-1} b] = \frac{\pi}{2} + \frac{\pi}{2} = \boldsymbol{\pi}$



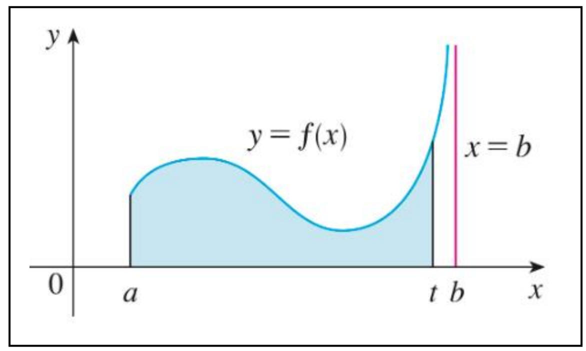
$$\int_{1}^{\infty} \frac{1}{x^p}\,dx \text{ is } \textbf{convergent} \text{ if } p > 1 \text{ and } \textbf{divergent} \text{ if } p \le 1$$

Discontinuous Integrands:

Definition of Improper Integral of Type 2:

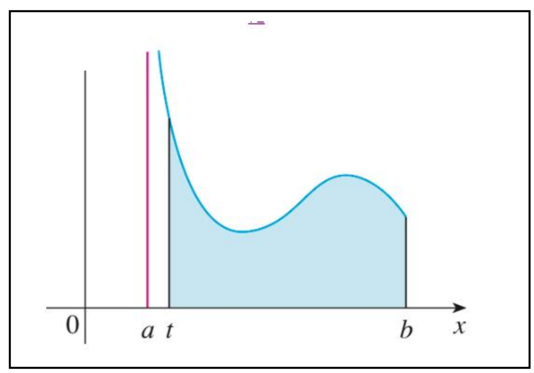
1. If ***f*** is continuous on [a, b) and is discontinuous at **b,**

Then $\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$ (if this limit exists as a finite number)



2. If ***f*** is continuous on (a, b] and is discontinuous at **a**,

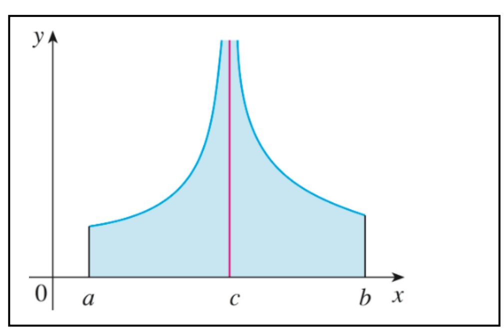
Then $\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$ (if this limit exists as a finite number)



For parts 1. and 2. the improper integral is called ***convergent*** if the corresponding limit exists and ***divergent*** if the limit does not exist.

3. If ***f*** is continuous on [a, b] except at **c**, where a < c < b, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are **convergent** (in others words the limits of the integrals exisit).

Then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$



If the limits in cases 1 – 2 exist, then the improper integral converges; otherwise, they diverge.

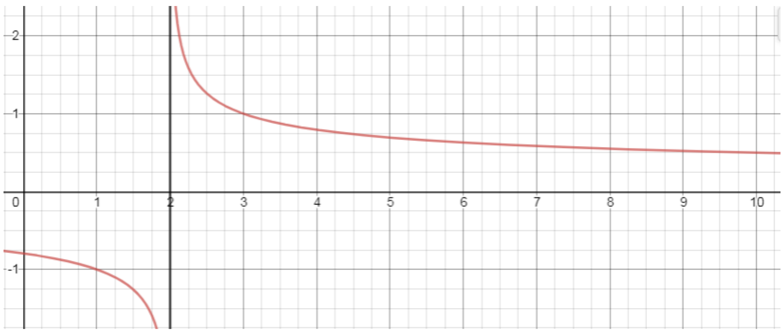
Example:

a) Evaluate

$$\int_1^{10} \frac{dx}{(x-2)^{\frac{1}{3}}}$$

Plot the function and you'll see the following:

The integrand is unbounded at **x = 2,** which appears to be an interior point of the interval of integration. We split the interval into two subintervals and evaluate an improper integral on each subinterval.



$$\int_1^{10} \frac{dx}{(x-2)^{\frac{1}{3}}} = \lim_{c\to 2-} \int_1^{c} \frac{dx}{(x-2)^{\frac{1}{3}}} + \lim_{d\to 2+} \int_d^{10} \frac{dx}{(x-2)^{\frac{1}{3}}}$$

$$= \lim_{c\to 2-} \frac{3}{2}(x-2)^{\frac{2}{3}}\Big|_1^c + \lim_{d\to 2+} \frac{3}{2}(x-2)^{\frac{2}{3}}\Big|_d^{10}$$

$$= \frac{3}{2}\left(\lim_{c\to 2-}\left[(c-2)^{\frac{2}{3}} - (1-2)^{\frac{2}{3}}\right]\right) + \frac{3}{2}\left(\lim_{d\to 2+}\left[(10-2)^{\frac{2}{3}} - (d-2)^{\frac{2}{3}}\right]\right)$$

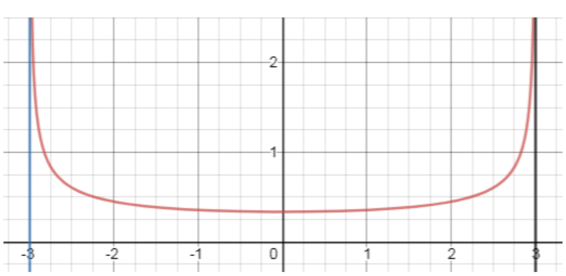
$$= \frac{3}{2}\left(0 - (-1)^{\frac{2}{3}}\right) + \frac{3}{2}\left((8)^{\frac{2}{3}} - 0\right)$$

$$= -\frac{3}{2} + \frac{3}{2}(4) = \mathbf{\frac{9}{2}}$$

b) Evaluate

$$\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx$$

Plot the function.



The integrand is even and has vertical asymptotes at x = ± 3.
Therefore,

$$\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx = 2\int_0^{3} \frac{1}{\sqrt{9-x^2}}\,dx$$

Because the improper integral is unbounded at **x = 3** we replace the upper limit with **c.**

$$2\int_0^3 \frac{1}{\sqrt{9-x^2}}dx = 2\lim_{c\to 3-}\int_0^c \frac{1}{\sqrt{9-x^2}} = 2\lim_{c\to 3-}\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\Big|_0^c$$

$$= 2\lim_{c\to 3-}\left(\sin^{-1}\left(\frac{c}{3}\right) - \sin^{-1}(0)\right) = 2\left(\frac{\pi}{2}\right) = \boldsymbol{\pi}$$

A comparison test for Improper Integrals:

Sometimes it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent. In such cases the following theorem is useful.

Comparison Theorem:

Suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuous functions with $\boldsymbol{f(x) \geq g(x) \geq 0 \;\; for\; x \geq a}$.

a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is convergent.

b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is divergent.

Example: Determine if the following integral diverges or converges.

$$\int_3^\infty \frac{1}{x+e^x}dx$$

Note that $\frac{1}{x+e^x} < \frac{1}{e^x}$ So if $\int_3^\infty \frac{1}{e^x}dx$ ***converges then*** $\int_3^\infty \frac{1}{x+e^x}dx$ ***will also converge***.

Find:

$$\int_3^\infty \frac{1}{e^x}dx = \lim_{b\to\infty}\int_3^b e^{-x}dx = \lim_{b\to\infty} -e^{-x}\Big|_3^b = \lim_{b\to\infty}(-e^{-b}+e^{-3}) = \boldsymbol{e^{-3}}$$

Since the limit exists, then $\int_3^\infty \boldsymbol{\frac{1}{e^x}dx}$ converges. Using the comparison theorem, $\int_3^\infty \boldsymbol{\frac{1}{x+e^x}dx}$ also converges.