More About Motion

In this section, we will consider the motion of a particle that is moving either "two-dimensionally" (in a *plane*) or "three-dimensionally" (in *space*). Its path is a curve, known as the *curve of motion*. This curve is defined either by a set of parametric equations or by a single vector equation. In either case, we use t as our parameter, representing time. Any particular value of t may be referred to as an "instant."

Two-Dimensional Motion

Motion along a curve can be described by a pair of parametric equations, $x = x(t), \ y = y(t)$. At any time t, the particle is located at the point $P_t = (x(t), y(t))$, whose position vector is $\mathbf{r}(t) = < x(t), y(t) > = x(t)\mathbf{i} + y(t)\mathbf{j}$. This vector equation is the particle's **position function**.

The particle's position at time $t = 0$ is $P_0 = (x(0), y(0))$, which is known as the particle's **initial position** or **starting point**.

The magnitude of $\mathbf{r}(t)$ is the **distance of the particle from the origin**. Note that $|\mathbf{r}(t)| = \sqrt{x(t)^2 + y(t)^2}$. This can be denoted $r(t)$. It is a scalar-valued function (whereas $\mathbf{r}(t)$ is a vector-valued function).

The particle's **velocity function** is $\mathbf{v}(t) = \mathbf{r}'(t) = < x'(t), y'(t) >$ or $< \frac{dx}{dt}, \frac{dy}{dt} >$ or $x'(t)\mathbf{i} + y'(t)\mathbf{j}$ or $\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$.

The particle's **speed function** is $v(t) = |\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. It is a scalar-valued function (whereas $\mathbf{v}(t)$ is a vector-valued function).

At any value of t, the particle is located at the point P_t. If, at this instant, the particle's speed is 0 (i.e., if its velocity is $\mathbf{0}$), then the curve of motion has a a **cusp** or **kink** or **sharp turn** at P_t. On the other hand, if the speed is not 0 (i.e., if its velocity is a nonzero vector), then the particle is said to have **smooth motion**, or to be **moving smoothly**, at this instant, and the curve of motion will have a **tangent line** at this point. In this context, the point P_t is referred to as the **point of tangency**. To write an equation for the tangent line, we use P_t as the initial point of the line and we use $\mathbf{v}(t)$ as the direction vector for the line. Bear in mind, this gives us a *directed line*, i.e., a line with a specified orientation. The **positive direction** of the tangent line is the direction of $\mathbf{v}(t)$, and the opposite direction is the negative direction. The direction of $\mathbf{v}(t)$ is known as the (instantaneous) **direction of motion** for the particle. Any nonzero vector having the same direction as $\mathbf{v}(t)$ may be referred to as a **tangent vector** for the particle at the point P_t. There are infinitely many such vectors, all of which are positive scalar multiples of each other. The velocity vector, $\mathbf{v}(t)$, is the unique tangent vector whose length is equal to the particle's speed, $v(t)$.

Recall that any nonzero vector divided by its own magnitude gives us a unit vector (i.e., a vector with length 1) having the same direction. Thus, if the velocity vector is nonzero, dividing it by the speed will give us a unit vector in the direction of motion. In other words, $\frac{\mathbf{v}(t)}{v(t)}$ is the unit vector in the direction of $\mathbf{v}(t)$. It is referred to as the **unit tangent vector** at the point P_t, and is denoted $\mathbf{T}(t)$. The unit tangent vector is *undefined* when the speed is zero.

At any value of t, if speed is nonzero, then $\mathbf{v}(t)$ and $\mathbf{T}(t)$ are nonzero vectors and can be represented as directed line segments. Although any nonzero vector may be placed at any starting point, we typically visualize $\mathbf{v}(t)$ and $\mathbf{T}(t)$ as being placed so that each has P_t is its tail. Then the tip of each vector will lie along the tangent line (in fact, the directed line segment for each vector will lie entirely along the tangent line). Both tips will lie on the same side (the positive side) of the tangent line. Since $\mathbf{T}(t)$ is a unit vector, its tip will lie a distance of exactly one unit from P_t. The distance between P_t and the tip of $\mathbf{v}(t)$ will of course be the speed, $v(t)$.

For nonzero speed, $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, i.e., the velocity vector is a scalar multiple of the unit tangent vector, where that scalar is the speed of motion.

The particle's **acceleration function** is $\mathbf{a}(t) = \mathbf{r}''(t) = < x''(t), y''(t) >$ or $< \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} >$ or $x''(t)\mathbf{i} + y''(t)\mathbf{j}$ or $\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$. Its magnitude is $a(t) = \sqrt{x''(t)^2 + y''(t)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$.

Before we proceed, let us summarize the formulas discussed so far. For brevity, we will omit "(t)". Also included are a few variations that will prove useful...

- $\mathbf{r} = < x, y >$
- $r = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}$
- $\frac{1}{r} = r^{-1} = (x^2 + y^2)^{-1/2}$
- $\mathbf{v} = < x', y' >$
- $v = \sqrt{x'^2 + y'^2} = (x'^2 + y'^2)^{1/2}$
- $\frac{1}{v} = v^{-1} = (x'^2 + y'^2)^{-1/2}$
- $v^2 = x'^2 + y'^2$
- $v^3 = (x'^2 + y'^2)^{3/2}$
- $\frac{1}{v^3} = v^{-3} = (x'^2 + y'^2)^{-3/2}$
- $\mathbf{T} = \frac{\mathbf{v}}{v} = \frac{1}{v}\mathbf{v} = (x'^2 + y'^2)^{-1/2} < x', y' >$
- $\mathbf{a} = < x'', y'' >$
- $a = \sqrt{x''^2 + y''^2} = (x''^2 + y''^2)^{1/2}$
- $\frac{1}{a} = a^{-1} = (x''^2 + y''^2)^{-1/2}$
- $\mathbf{v} \cdot \mathbf{a} = x'x'' + y'y''$

$\mathbf{T}(t)$ is a vector-valued function of time t, but it is always a vector with a *fixed* length (namely, length one). Hence, as time varies, the only thing about $\mathbf{T}(t)$ that can change is its *direction*. The *rate* at which our particle changes direction is found by differentiating $\mathbf{T}(t)$ with respect to time, in other words, by finding $\mathbf{T}'(t)$. But bear in mind, the derivative of a vector-valued function is anothe vector-valued function. If we wish to express the rate of direction change as a *scalar*, then we compute the *magnitude* of $\mathbf{T}'(t)$.

Since $\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t)$, we apply the Product Rule, giving us
$\mathbf{T}'(t) = \left(\frac{1}{v(t)}\right)'\mathbf{v}(t) + \frac{1}{v(t)}\mathbf{v}'(t) = \left(\frac{1}{v(t)}\right)'\mathbf{v}(t) + \frac{1}{v(t)}\mathbf{a}(t).$

Since $\frac{1}{v(t)} = (x'(t)^2 + y'(t)^2)^{-1/2}$, we apply the Chain Rule, giving us
$\left(\frac{1}{v(t)}\right)' = -\frac{1}{2}(x'(t)^2 + y'(t)^2)^{-3/2}(2x'(t)x''(t) + 2y'(t)y''(t)) =$
$-(x'(t)^2 + y'(t)^2)^{-3/2}(x'(t)x''(t) + y'(t)y''(t)) =$
$-v(t)^{-3}(x'(t)x''(t) + y'(t)y''(t)).$

Now we have $\mathbf{T}'(t) = -v(t)^{-3}(x'(t)x''(t) + y'(t)y''(t))\mathbf{v}(t) + v(t)^{-1}\mathbf{a}(t)$
$= v(t)^{-1}\mathbf{a}(t) - v(t)^{-3}(x'(t)x''(t) + y'(t)y''(t))\mathbf{v}(t)$
$= (x'(t)^2 + y'(t)^2)^{-3/2}[(x'(t)^2 + y'(t)^2)\mathbf{a}(t) - (x'(t)x''(t) + y'(t)y''(t))\mathbf{v}(t)]$

Since $v(t) = \sqrt{x'(t)^2 + y'(t)^2}$, $v(t)^2 = x'(t)^2 + y'(t)^2$. Thus, we may write
$\mathbf{T}'(t) = (v(t)^2)^{-3/2}[v(t)^2\mathbf{a}(t) - (x'(t)x''(t) + y'(t)y''(t))\mathbf{v}(t)]$,
which simplifies to $\mathbf{T}'(t) = v(t)^{-3}[v(t)^2\mathbf{a}(t) - (x'(t)x''(t) + y'(t)y''(t))\mathbf{v}(t)]$
For brevity, we may write $\mathbf{T}' = v^{-3}[v^2\mathbf{a} - (x'x'' + y'y'')\mathbf{v}]$,
or we may write $\mathbf{T}' = \frac{v^2\mathbf{a} - \left(x'x'' + y'y''\right)\mathbf{v}}{v^3}$

Note that $x'x'' + y'y'' = \mathbf{v} \cdot \mathbf{a}$. Thus, $\mathbf{T}' = \frac{v^2\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{v^3}$

Since $v^2 = \mathbf{v} \cdot \mathbf{v}$, we could also write $\mathbf{T}' = \frac{(\mathbf{v}\cdot\mathbf{v})\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{v^3}$

When computing the magnitude of a vector, a positive scalar coefficient will factor out. Thus,
$|\mathbf{T}'| = \frac{1}{v^3}\left|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\right|$, or $(x'(t)^2 + y'(t)^2)^{-3/2}\left|(\mathbf{v} \cdot \mathbf{v})\mathbf{a} - (\mathbf{v} \cdot \mathbf{a})\mathbf{v}\right|$.

To illustrate, suppose a particle is moving along the parabola $y = x^2$ with position function $\mathbf{r}(t) = < t, t^2 >$. Then:
$r(t) = \sqrt{t^2 + t^4}$
$\mathbf{v}(t) = < 1, 2t >$
$v(t) = \sqrt{1 + 4t^2} = (1 + 4t^2)^{1/2}$
$\mathbf{a}(t) = < 0, 2 >$
$\mathbf{T}(t) = (1 + 4t^2)^{-1/2} < 1, 2t > = \frac{<1,2t>}{\sqrt{1+4t^2}}$
$v(t)^2 = 1 + 4t^2$, $v(t)^3 = (1 + 4t^2)^{3/2}$, and $x'x'' + y'y'' = (1)(0) + (2t)(2) = 4t$, so
$\mathbf{T}'(t) = \frac{(1+4t^2)\mathbf{a} - 4t\mathbf{v}}{(1+4t^2)^{3/2}}$
The numerator is $(1 + 4t^2) < 0, 2 > - 4t < 1, 2t > = < 0, 2 + 8t^2 > - < 4t, 8t^2 > = < -4t, 2 >$,

so $\mathbf{T}'(t) = \frac{<-4t,2>}{(1+4t^2)^{3/2}}$ or $(1+4t^2)^{-3/2} < -4t, 2 >$

$|\mathbf{T}'(t)| = (1+4t^2)^{-3/2}|< -4t, 2 >| = (1+4t^2)^{-3/2}\sqrt{16t^2+4} = 2(1+4t^2)^{-3/2}\sqrt{1+4t^2} = \frac{2}{1+4t^2}$

Three-Dimensional Motion

Motion along a curve can be described by three parametric equations, $x = x(t)$, $y = y(t)$, $z = z(t)$. At any time t, the particle is located at the point $P_t = (x(t), y(t), z(t))$, whose position vector is $\mathbf{r}(t) = < x(t), y(t), z(t) > = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. This vector equation is particle's **position function**.

The particle's **initial position** is $P_0 = (x(0), y(0), z(0))$.

The particle's **distance from the origin** is $r(t) = |\mathbf{r}(t)| = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$.

The particle's **velocity function** is $\mathbf{v}(t) = \mathbf{r}'(t) = < x'(t), y'(t), z'(t) >$ or $< \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >$ or $x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ or $\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$.

The particle's **speed function** is

$v(t) = |\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$.

For nonzero speed, the **unit tangent vector** is $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)}$.

For nonzero speed, $\mathbf{v}(t) = v(t)\mathbf{T}(t)$.

The particle's **acceleration function** is $\mathbf{a}(t) = \mathbf{r}''(t) = < x''(t), y''(t), z''(t) >$ or $< \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} >$ or $x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ or $\frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$. Its magnitude is $a(t) = \sqrt{x''(t)^2 + y''(t)^2 + z''(t)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$.

$\mathbf{T}'(t) =$
$(x'(t)^2 + y'(t)^2 + z'(t)^2)^{-3/2}[(x'(t)^2 + y'(t)^2 + z'(t)^2)\mathbf{a}(t) - (x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t))\mathbf{v}(t)]$
which simplifies to $\mathbf{T}'(t) = v(t)^{-3}[v(t)^2\mathbf{a}(t) - (x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t))\mathbf{v}(t)]$
For brevity, we may write $\mathbf{T}' = v^{-3}[v^2\mathbf{a} - (x'x'' + y'y'' + z'z'')\mathbf{v}]$,
or we may write $\mathbf{T}' = \frac{v^2\mathbf{a} - \left(x'x'' + y'y'' + z'z''\right)\mathbf{v}}{v^3}$