3.1 Derivatives of Polynomials and Exponential Functions

In this section we will learn how to differentiate constant functions, power functions, polynomials, and exponential functions.

Let's start with the constant function ***f(x) = C.*** Remember that the graph of this function is a horizontal line ***y = C,*** which has a slope of ***0.*** Therefore ***f'(x)*** must equal 0. Using the definition this is what it looks like.

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{C-C}{h} = \lim_{h\to 0} 0 = 0$ (The limit of a constant = constant.)

Derivative of a Constant: $\frac{d}{dx}C = 0$

Power functions: Let's analyze the function ***y = x.*** From the graph of this function , we know that the slope of ***y = x*** is equal to 1. Using the definition of the derivative we can prove this.

If ***f(x) = x,*** then $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{x+h-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$

Derivative of the function *f(x) = x* or (y = x): $\frac{d}{dx}(x) = 1$

Example: Use the definition of the derivative to find $f'(x)$ if

a) $f(x) = x^2$
b) $f(x) = x^3$

a) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0} 2x+h$

$$\boldsymbol{f'(x) = 2x}$$

b) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$

$= \lim_{h\to 0} 3x^2+3xh+h^2$ $\qquad$ $\boldsymbol{f'(x) = 3x^2}$

Notice that there is a pattern; as long as n > 0, and n is an integer, then we get the following:

The Power Rule: If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example: Use the power rule to find the derivative of the following fuctions:

a) $f(x) = x^8$ $\qquad$ $f'(x) = 8x^7$

b) $y = x^{100}$ $\quad\quad y' = 100x^{99}$

c) $f(x) = 2x^{10}$ $\quad\quad f'(x) = 20x^9$

Notice that the example above only took into account of $\boldsymbol{n}$ being a positive integer, but the power rule actually works for any real number $\boldsymbol{n}$. Therefore we get the following:

The Power Rule (General Version): If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example: Use the Power Rule to find the derivative of the following functions:

a) $f(x) = \frac{1}{x^3}$ Rewrite $\frac{1}{x^3}$ as x^{-3} where n = -3. Then use the power rule. $\boldsymbol{f'(x) = -3x^{-4}}$ or $\boldsymbol{f'(x) = -\frac{3}{x^4}}$

b) $f(x) = \sqrt{x}$ Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ where n = $\frac{1}{2}$. $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad \boldsymbol{f'(x) = \frac{1}{2\sqrt{x}}}$

New Derivative from Old:

When new functions are created by simply adding, subtracting or multiplying by a constant to an old function, then their derivatives can be calculated in terms of the derivative of the old functions.

The Constant Multiple Rule: If $\boldsymbol{c}$ is a constant and $\boldsymbol{f}$ is a differentiable function, then:

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$

The Sum Rule: If $\boldsymbol{f}$ and $\boldsymbol{g}$ are both differentiable, then:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule: If $\boldsymbol{f}$ and $\boldsymbol{g}$ are both differentiable, then:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

We will complete one of these proofs in class. If you are interested in seeing them all, they are on pages 175 – 176 in your textbook.

Example: Use the constant multiple rule, the sum rule and the difference rule to find the following derivative: $\quad \boldsymbol{f(x) = \frac{7}{4}x^2 - 3x + 12}$

$$f'(x) = \frac{d}{dx}\left[\frac{7}{4}x^2 - 3x + 12\right] = \frac{d}{dx}\left(\frac{7}{4}x^2\right) - \frac{d}{dx}(3x) + \frac{d}{dx}(12) = \frac{7}{4}\frac{d}{dx}(x^2) - 3\frac{d}{dx}(x) + \frac{d}{dx}(12)$$

$f'(x) = \frac{7}{4} \cdot 2x - 3 \cdot 1 + 0 \quad\quad \boldsymbol{f'(x) = \frac{7}{2}x - 3}$

Exponential Functions:

If we let the exponential function be $f(x) = b^x$, we can try to compute the derivative by using the definition of a derivative.

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} \frac{b^x b^h - b^x}{h} = \lim_{h\to 0} \frac{b^x(b^h-1)}{h}$ (Notice that b^x doesn't depend on h, so we can pull it out in front of the limit.)

$f'(x) = b^x \lim_{h\to 0} \frac{b^h-1}{h}$ If we let x = 0, then $b^x = 1$. We see that the limit is the value of the derivative of $\boldsymbol{f}$ at 0, in other words,

$\lim_{h\to 0} \frac{b^h-1}{h} = f'(0)$ therefore, $f'(x) = b^x \cdot f'(0)$ (A more exact derivative rule for exponential functions is given later in this chapter. But this leads us into our next definition.)

Definition of the number *e*.

e is a number such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$

Now using $f'(x) = b^x \cdot f'(0)$ and let $b = e$ and $f'(0) = 1$, then we get the following:

Derivative of the Natural Exponential Function: $\boldsymbol{\frac{d}{dx}[e^x] = e^x}$

This shows that the exponential function $f(x) = e^x$ has the property that it is its own derivative.

Example: If $f(x) = 2e^x$, find $f'(x)$ and if $g(x) = e^2$, find $g'(x)$.

$f'(x) = \frac{d}{dx}(2e^x)$

$f'(x) = 2\frac{d}{dx}(e^x)$

$f'(x) = 2e^x$

$g'(x) = \frac{d}{dx}(e^2)$

$g'(x) = e^2\frac{d}{dx}(1)$

$g'(x) = e^2 \cdot 0$

$g'(x) = 0$