# 15.1-15.2) Double Integrals

A *rectangular region* in the $x,y$ plane is a region $R = \{(x,y) \mid x \in [a,b],\ y \in [c,d]\} = [a,b] \times [c,d]$, where $a < b$ and $c < d$. Technically, this is a closed region bounded by a rectangle. Casually speaking, we may refer to this region as a “rectangle” (but bear in mind, we mean to include the *interior* of the rectangle, as well as the rectangle itself).

A double integral over rectangle $R$ is $\iint_R f(x,y)\ dA$. This may be *iterated* in two possible ways (Fubini’s Theorem guarantees these are equivalent, so long as $f$ is continuous on $R$):

- $\int_a^b \int_c^d f(x,y)\ dy\ dx$
- $\int_c^d \int_a^b f(x,y)\ dx\ dy$

**Example One**: Let $R = [1,2] \times [0,\pi]$. $\iint_D y\sin(xy)\ dA$ might be iterated as either $\int_1^2 \int_0^\pi y\sin(xy)\ dy\ dx$ or $\int_0^\pi \int_1^2 y\sin(xy)\ dx\ dy$. Which iteration should we choose? Clearly the second way is easier, since the first way would require integration by parts (twice!!). Note that the second way can also be written as $\int_0^\pi y \int_1^2 \sin(xy)\ dx\ dy$.

$$\int_1^2 \sin(xy)\ dx = [-\tfrac{1}{y}\cos(xy)]_{x=1}^{x=2} = \tfrac{1}{y}[\cos(xy)]_{x=2}^{x=1} = \tfrac{1}{y}(\cos y - \cos 2y)$$

Thus we get $\int_0^\pi y \cdot \tfrac{1}{y}(\cos y - \cos 2y)\ dy = \int_0^\pi (\cos y - \cos 2y)\ dy = \left[\sin y - \tfrac{1}{2}\sin 2y\right]_0^\pi = 0$

**The Integral Factorization Principle**: Suppose $f(x,y)$ can be factored into a product of two functions, one depending only on $x$ and the other depending only on $y$, say $r(x)s(y)$. Then $\int_a^b \int_c^d f(x,y)\ dy\ dx = \int_a^b \int_c^d r(x)s(y)\ dy\ dx = \int_a^b r(x) \int_c^d s(y)\ dy\ dx = \int_c^d s(y)\ dy \int_a^b r(x)\ dx$, or $\int_a^b r(x)\ dx \int_c^d s(y)\ dy$.

**Example Two**: $\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 x^2 \, dx \int_1^2 y \, dy = \left[ \frac{1}{3}x^3 \right]_0^3 \left[ \frac{1}{2}y^2 \right]_1^2 =$
$\frac{1}{3}(27-0)\frac{1}{2}(4-1) = (9)(\frac{3}{2}) = \frac{27}{2}$

A *Type I region* in the $x,y$ plane is a region $D = \{(x,y) \mid x \in [a,b],\ g_1(x) \le y \le g_2(x)\}$. If $D$ is a Type I region, then $\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$. This is a *Type I double integral*.

A *Type II region* in the $x,y$ plane is a region $D = \{(x,y) \mid y \in [c,d],\ h_1(y) \le x \le h_2(y)\}$. If $D$ is a Type II region, then $\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$. This is a *Type II double integral*.

Fubini's Theorem does *not* apply to double integrals over Type I or II regions; we *cannot* rewrite $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$ as $\int_{g_1(x)}^{g_2(x)} \int_a^b f(x,y)\, dx\, dy$, and we *cannot* rewrite $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$ as $\int_{h_1(y)}^{h_2(y)} \int_c^d f(x,y)\, dy\, dx$.

**Example Three**: Let $D$ be the region bounded by the parabolas $y = 2x^2$ and $y = x^2 + 1$, which intersect at the points $(-1,2)$ and $(1,2)$. This is a Type I region.

$\iint_D (x+2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{x^2+1} (x+2y)\, dy\, dx.$

$\int_{2x^2}^{x^2+1} (x+2y)\, dy = [xy + y^2]_{y=2x^2}^{y=x^2+1} = (x(x^2+1) + (x^2+1)^2) - (x(2x^2) + (2x^2)^2) =$
$(x^3 + x + x^4 + 2x^2 + 1) - (2x^3 + 4x^4) =$
$x^3 + x + x^4 + 2x^2 + 1 - 2x^3 - 4x^4 =$
$-3x^4 - x^3 + 2x^2 + x + 1$

So we get $\int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1) dx = \left[ -\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right]_{-1}^{1} =$
$(-\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1) - (\frac{3}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1) =$
$-\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1 - \frac{3}{5} + \frac{1}{4} + \frac{2}{3} - \frac{1}{2} + 1 =$
$-\frac{6}{5} + \frac{4}{3} + 2 = -\frac{18}{15} + \frac{20}{15} + \frac{30}{15} = \frac{32}{15}.$

Sometimes it is necessary to reiterate a Type I integral as Type II, or vice versa...

**Example Four**: The double integral $\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx$ is Type I.

$D = \{(x,y)\ |\ x \in [0,1],\ x \le y \le 1\}$. The lower boundary of $D$ is the line $y = x$ and the upper boundary is the line $y = 1$, and these lines intersect at the point $(1,1)$. $D$ also has the vertical boundary line $x = 0$. $D$ is thus a right triangle with vertices $(0,0)$, $(0,1)$, and $(1,1)$, with the right angle at $(0,1)$.

The problem is that, with the integral iterated this way, we are unable to antidifferentiate–the antiderivative of $\sin(y^2)$ is a non-elementary function. To circumvent this problem, we reiterate the integral as a Type II integral.

When we look at $D$ as a Type II region, the "lower" boundary is the line $x = 0$ and the "upper" boundary is the line $x = y$. Thus, from a Type II perspective, $D =$ $\{(x,y)\ |\ y \in [0,1],\ 0 \le x \le y\}$.

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\ dx = \int_0^1 \int_0^y \sin(y^2)\ dx\ dy = \int_0^1 \sin(y^2) \int_0^y dx\ dy$$

$\int_0^y dx = [x]_0^y = y$, so we obtain $\int_0^1 \sin(y^2)\ y\ dy$

We can evaluate this with an ordinary Calculus I substitution. Let $u = y^2$ so $du = 2y\ dy$, and $\frac{1}{2}du = y\ dy$. When $y = 0,\ u = 0$, and when $y = 1,\ u = 1$. Hence, $\int_0^1 \sin(y^2)\ y\ dy =$

$$\int_0^1 \sin u\ \tfrac{1}{2}du = \tfrac{1}{2}\int_0^1 \sin u\ du = \tfrac{1}{2}[-\cos u]_0^1 = \tfrac{1}{2}[\cos u]_1^0 = \tfrac{1}{2}(\cos 0 - \cos 1) = \tfrac{1}{2}(1 - \cos 1)$$

We aren't always given an integration problem where the integral is already set up–sometimes we have to set up the integral ourselves. In such cases, the challenge is to figure out the region in the $x,y$ plane over which we are integrating. If we are lucky, the region will be rectangular, but otherwise we must figure out whether to analyze the region as Type I or Type II.

**Example Five**: Find the volume of the solid bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes. *Solution:* The three coordinate planes are $z = 0$, which is the $x,y$ plane, $y = 0$, which is the $x,z$ plane, and $x = 0$, which is the $y,z$ plane. The region of integration in the $x,y$ plane is the rectangle bounded by the vertical lines $x = 0$ and $x = 2$, and by the horizontal lines $y = 0$ and $y = 2$, i.e., $R = [0,2] \times [0,2]$. Here $f(x) = 16 - x^2 - 2y^2$, so the volume is $\int_0^2 \int_0^2 (16 - x^2 - 2y^2)\ dy\ dx$.

When we evaluate this integral, we get $48$.

**Example Six**: Find the volume of a tetrahedron (i.e., a pyramid with a triangular base) bounded by the planes $x+2y+z=2$, $y=\frac{1}{2}x$, $x=0$, and $z=0$. *Solution:* Notice that $z=0$ is a horizontal plane (i.e., it is the $x,y$ plane), $y=\frac{1}{2}x$ and $x=0$ are vertical planes (the latter is the $y,z$ plane), and $x+2y+z=2$ is an oblique plane (it has $x$ intercept $2$, $y$ intercept $1$, and $z$ intercept $2$). The intersection of the planes $z=0$ and $x=0$ is the $y$ axis, the intersection of the planes $z=0$ and $y=\frac{1}{2}x$ is the line $y=\frac{1}{2}x$, and the intersection of the planes $z=0$ and $x+2y+z=2$ is the line $x+2y=2$, i.e., $y=-\frac{1}{2}x+1$. $D$ is the region in the $x,y$ plane bounded by the $y$ axis and by the lines $y=\frac{1}{2}x$ and $y=-\frac{1}{2}x+1$. This is a triangular region with vertices $(0,0)$, $(0,1)$, and $(1,\frac{1}{2})$. This is best viewed as a Type I region. Here, $f(x)=2-x-2y$, so the volume is $\int_0^1 \int_{\frac{1}{2}x}^{-\frac{1}{2}x+1} (2-x-2y)\ dy\ dx$. When we evaluate this integral, we get $\frac{1}{3}$.

The Integral Factorization Principle does *not* apply to double integrals over Type I or II regions; we *cannot* rewrite $\int_a^b \int_{g_1(x)}^{g_2(x)} r(x)s(y)\ dy\ dx$ as $\int_a^b r(x)\ dx \int_{g_1(x)}^{g_2(x)} s(y)\ dy$, and we *cannot* rewrite $\int_c^d \int_{h_1(y)}^{h_2(y)} r(x)s(y)\ dx\ dy$ as $\int_c^d s(y)\ dy \int_{h_1(y)}^{h_2(y)} r(x)\ dx$. However, we *can* write the former integral as $\int_a^b r(x) \int_{g_1(x)}^{g_2(x)} s(y)\ dy\ dx$, and we *can* write the latter integral as $\int_c^d s(y) \int_{h_1(y)}^{h_2(y)} r(x)\ dx\ dy$. (We cannot factor the inner integral out of the outer integral, because the boundaries of integration of the inner integral involve the variable of integration for the outer integral.)

1. If $S(y)$ is an antiderivative of $s(y)$ with respect to $y$, then $\int_{g_1(x)}^{g_2(x)} s(y)\ dy = [S(y)]_{g_1(x)}^{g_2(x)} = S(g_2(x)) - S(g_1(x))$, and so $\int_a^b r(x) \int_{g_1(x)}^{g_2(x)} s(y)\ dy\ dx = \int_a^b r(x)[S(g_2(x)) - S(g_1(x))]dx$.

2. If $R(y)$ is an antiderivative of $r(y)$ with respect to $y$, then $\int_{h_1(y)}^{h_2(y)} r(x)\ dx = [R(x)]_{h_1(y)}^{h_2(y)} = R(h_2(y)) - R(h_1(y))$, and so $\int_c^d s(y) \int_{h_1(y)}^{h_2(y)} r(x)\ dx\ dy = \int_c^d s(y)[R(h_2(y)) - R(h_1(y))]dy$.

**Example Seven**: $\int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy\ dx\ dy = \int_{-2}^{4} y \int_{\frac{1}{2}y^2-3}^{y+1} x\ dx\ dy$. Let us work this problem using formula 2. Here, $r(x)=x$ and $s(y)=y$. $R(x)=\frac{1}{2}x^2$, so $R(y+1)=\frac{1}{2}(y+1)^2 =$

$\frac{1}{2}(y^2+2y+1)$, and $R(\frac{1}{2}y^2-3) = \frac{1}{2}(\frac{1}{2}y^2-3)^2 = \frac{1}{2}(\frac{1}{4}y^4-3y^2+9)$. Hence, we obtain

$$\int_{-2}^{4} y\left[\tfrac{1}{2}(y^2+2y+1)-\tfrac{1}{2}(\tfrac{1}{4}y^4-3y^2+9)\right]dy = \tfrac{1}{2}\int_{-2}^{4} y\left(y^2+2y+1-\tfrac{1}{4}y^4+3y^2-9\right)dy =$$

$$\tfrac{1}{2}\int_{-2}^{4} y\left(4y^2+2y-\tfrac{1}{4}y^4-8\right)dy = \tfrac{1}{2}\int_{-2}^{4}\left(4y^3+2y^2-\tfrac{1}{4}y^5-8y\right)dy =$$

$$\int_{-2}^{4}\left(2y^3+y^2-\tfrac{1}{8}y^5-4y\right)dy = \left[\tfrac{1}{2}y^4+\tfrac{1}{3}y^3-\tfrac{1}{48}y^6-2y^2\right]_{-2}^{4} = 36.$$