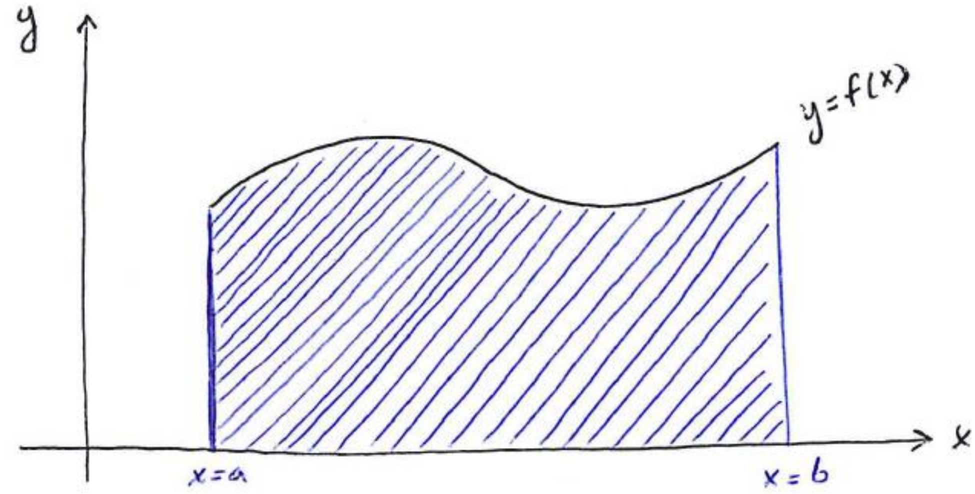
5.1 Area and Distances

In this section we will learn how to approximate the area under a curve or the distance traveled by a car.

The Area Problem

Say that we have a function $y = f(x)$ that is continuous from a to b. If we would like to find the area under this curve $y = f(x)$, which is the shaded region, we must bound the graph of f with vertical lines at $x = a$ and $x = b$, and the x-axis. See the figure below.



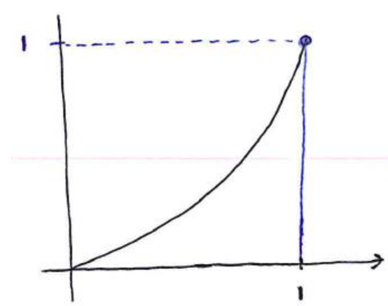
But notice that this is a difficult problem because it's not easy to find the area of a region that has curved sides. In other words, if we have a horizontal line or a line with a constant slope instead of the curved side, this would not be that difficult of a problem because we could use the formula for the area of a rectangle or a triangle to find the area.

With these problems we can approximate the area of the regions by using rectangles. If we break up the region into many rectangles, we will have a better approximation of the region. Whereas, if we use less rectangles, the approximation will not be as good.

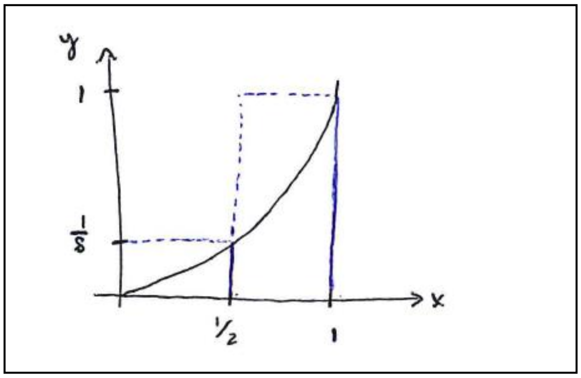
Example: Use rectangles to estimate the area under the curve of $y = x^3$ from 0 to 1.

Let's complete this task by breaking up the area into 1, 2 and 4 rectangles.

For 1 rectangle: The area of $y = x^3$ from 0 to 1, $A = (1)(1)^3 = 1$ (Using the height from the right side of the rectangle.) Remember the area of a rectangle is base x height.

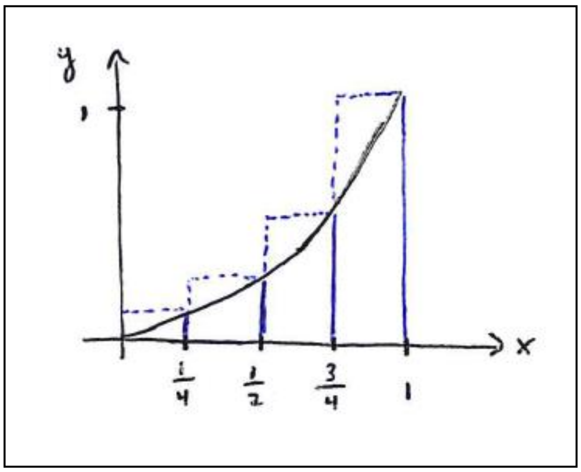


For 2 rectangles:



$$A = A_{0-\frac{1}{2}} + A_{\frac{1}{2}-1}$$
$$A = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)(1)^3$$
$$A = \frac{1}{16} + \frac{1}{2}$$
$$A = \frac{9}{16} = .5625$$

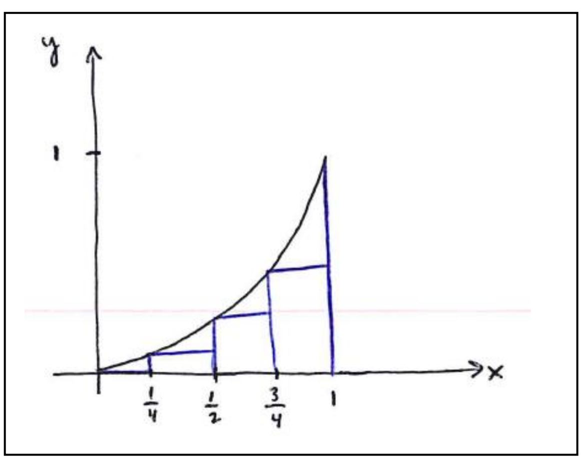
For 4 rectangles:



$$A = A_{0-\frac{1}{4}} + A_{\frac{1}{4}-\frac{1}{2}} + A_{\frac{1}{2}-\frac{3}{4}} + A_{\frac{3}{4}-1}$$
$$A = \tfrac{1}{4}\left(\tfrac{1}{4}\right)^3 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^3 + \tfrac{1}{4}(1)^3$$
$$A = \frac{1}{256} + \frac{1}{32} + \frac{27}{256} + \frac{1}{4}$$
$$A = \frac{1+8+27+64}{256} = \frac{100}{256} = \frac{25}{64} = .390625$$

Notice that the area of $y = x^3$ seems to be getting smaller as we increase the number of rectangles. For this reason we could say that the area of $y = x^3$ from $x = 0$ to $x = 1$ is less than **.390625**. In the example we just completed the **x-values** used were the **right end-points** of every rectangle. Let's denote the area found for the curve $y = x^3$ from $x = 0$ to $x = 1$ using the right end-points as **A_R**. So we can say that **$A_R < .390625$.**

Now instead of using the **right end-points** of each rectangle, let's use the **left end-points** of each rectangle to find the area under the curve $y = x^3$ from $x = 0$ to $x = 1$. Denote this area as **A_L**. We will also use 4 rectangles.



$$A_L = A_{0-\frac{1}{4}} + A_{\frac{1}{4}-\frac{1}{2}} + A_{\frac{1}{2}-\frac{3}{4}} + A_{\frac{3}{4}-1}$$
$$A_L = \tfrac{1}{4}(0)^3 + \tfrac{1}{4}\left(\tfrac{1}{4}\right)^3 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^3 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^3$$
$$A_L = 0 + \tfrac{1}{4}\left(\tfrac{1}{64}\right) + \tfrac{1}{4}\left(\tfrac{1}{8}\right) + \tfrac{1}{4}\left(\tfrac{27}{64}\right)$$
$$A_L = \frac{1}{256} + \frac{1}{32} + \frac{27}{256}$$
$$A_L = \frac{1+8+27}{256} = \frac{36}{256} = \frac{9}{64} = .140625$$

Using these two methods to find the area, we can see that the **actual area, A,** is between the left and right estimations. In other words $\boldsymbol{A_L < A < A_R}$

$$.140625 < A < .390625$$

If we were to use 16 intervals (rectangles) we would get $A_R \approx 0.2822$ and $A_L \approx 0.2197$ therefore, $0.2197 < A < 0.2822$.

We could get better approximations by increasing the number of rectangles. I have created a table to show the results using $\boldsymbol{n}$ number of rectangles using the **left** and **right** endpoints to calculate the heights.

With 1000 rectangles we can see that **A** is between $\mathbf{A_L}$ and $\mathbf{A_R}$. We could say that **A = .25** (The average of A_L and A_R) This is a good estimate for the area under the curve of $\boldsymbol{y = x^3}$ from $\boldsymbol{x = 0}$ to $\boldsymbol{x = 1}$.

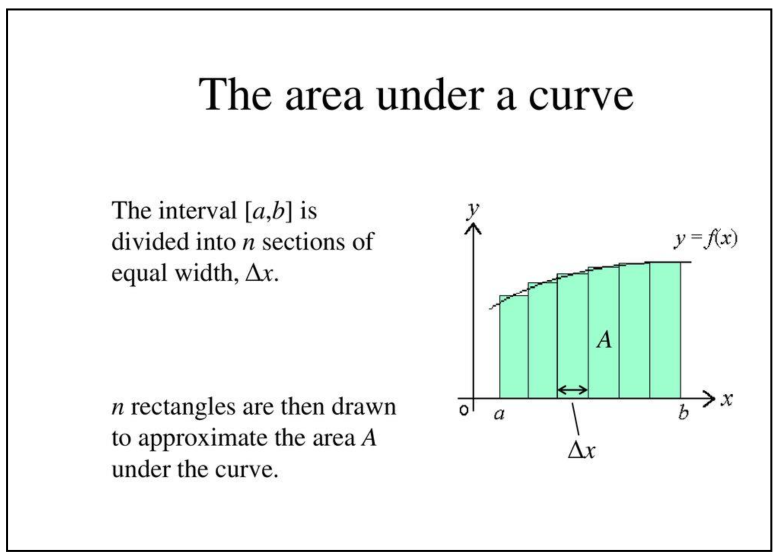
It can be proved that $\mathbf{A_L = A_R}$ if we have $n \to \infty$.

n	A_L	A_R
20	.2256	.2756
50	.2401	.2601
100	.2450	.2550
200	.2475	.2525
500	.2490	.2510
1000	.2495	.2505

Now let's take this idea a step further and subdivide a region into **n** rectangles of equal width as show below.

The width of the interval [a, b] is b – a, so the Width of each of the **n** rectangles is $\Delta x = \frac{b-a}{n}$.
The rectangles divide the interval [a, b] into **n** subintervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$
Where $\mathbf{a = x_0}$ and $\mathbf{b = x_n}$. The right endpoints of the subintervals are

$x_1 = a + \Delta x$
$x_2 = a + 2\Delta x$
$x_3 = a + 3\Delta x$
$\vdots$



Let's approximate the $\boldsymbol{i^{th}}$ rectangle, $\boldsymbol{A_i}$ by a rectangle with width $\boldsymbol{\Delta x}$ and height $\boldsymbol{f(x_i)}$, which is the value of $\boldsymbol{f}$ at the right endpoint. Then the area of the $\boldsymbol{i^{th}}$ rectangle is $\boldsymbol{f(x_i) \cdot \Delta x}$. Using this idea, we can approximate the area of the region by find the sum of the areas of the rectangles, that is
$\boldsymbol{R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x}$ (where R_n represents the area of the region using right endpoints)

Definition: The area **A** of the region that lies under the graph of the continuous function $\boldsymbol{f}$ is the **limit** of the **sum** of the areas of the approximating rectangles:

$\boldsymbol{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x]}$.

It can be proved that we get the same value if we use left endpoints:

$\boldsymbol{A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x]}$ ($\boldsymbol{L_n}$ represents the area of the region using left endpoints)

We often use sigma notation ($\boldsymbol{\Sigma}$) to write sums. For example,

$\sum_{i=1}^{n} f(x_i) = \boldsymbol{f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x}$

Therefore, the expressions for area using the **left** and **right** endpoints are:

$$A_L = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A_R = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

The following theorem might be useful when completing some of these summations.

Theorem: Sums of Powers of Integers.

Let **n** be a positive integer and **c** be a real number.

THEOREM 4.2 Summation Formulas

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example: Use the definition to find an expression for the area under the graph of **f** as a limit. Do not evaluate the limit. Use the right endpoints. $\boldsymbol{f(x) = \frac{2x}{x^2+1}}$, $\boldsymbol{1 \le x \le 3}$

Since **a = 1** and **b = 3**, then $\boldsymbol{\Delta x = \frac{3-1}{n} = \frac{2}{n}}$ so $x_1 = 1 + \frac{2}{n}, x_2 = 1 + 2 \cdot \frac{2}{n}, x_{32} = 1 + 3 \cdot \frac{2}{n}, \ldots, x_n = 1 + n \cdot \frac{2}{n}$

The sum of the areas of the approximating rectangles is

$$\boldsymbol{R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x}$$

$$R_n = \frac{2x_1}{(x_1)^2+1}\Delta x + \frac{2x_2}{(x_2)^2+1}\Delta x + \frac{2x_3}{(x_3)^2+1}\Delta x + \cdots + \frac{2x_n}{(x_n)^2+1}$$

$$R_n = \frac{2\left(1+\frac{2}{n}\right)}{\left(1+\frac{2}{n}\right)^2+1}\left(\frac{2}{n}\right) + \frac{2\left(1+\frac{4}{n}\right)}{\left(1+\frac{4}{n}\right)^2+1}\left(\frac{2}{n}\right) + \frac{2\left(1+\frac{6}{n}\right)}{\left(1+\frac{6}{n}\right)^2+1}\left(\frac{2}{n}\right) + \cdots + \frac{2\left(1+\frac{2n}{n}\right)}{\left(1+\frac{2n}{n}\right)^2+1}\left(\frac{2}{n}\right)$$

Using the definition . . .

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}\left(\frac{2}{n}\right)$$

$$\boxed{A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{2\left(1+\frac{2i}{n}\right)}{\left(1+\frac{2i}{n}\right)^2+1}}$$

Midpoint

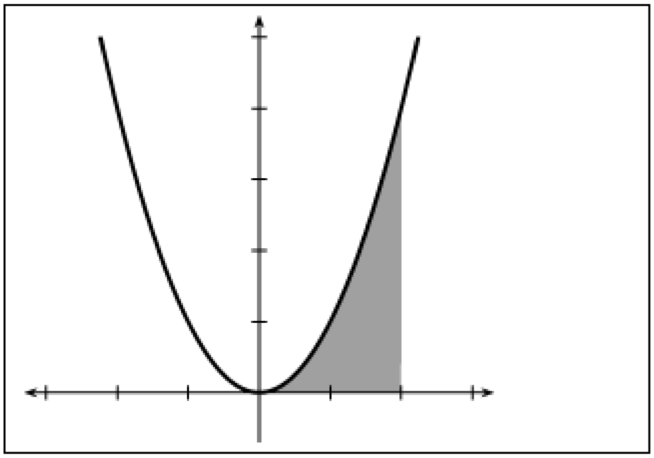
What if instead of using the left endpoints or the right endpoints we instead used the **midpoints** of the subintervals. Suppose $\boldsymbol{f}$ is defined on a closed interval [a, b], which is divided into **n** subintervals of equal width, **Δx.** If $\boldsymbol{x_i}$ is a point in the $\boldsymbol{i^{th}}$subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots$, n, then the area under the curve would be $\boldsymbol{A} = \mathbf{lim}_{\boldsymbol{n}\to\infty} \boldsymbol{M_n} = \mathbf{lim}_{\boldsymbol{n}\to\infty} \boldsymbol{f}(\boldsymbol{x_i})\Delta\boldsymbol{x}$ where $\boldsymbol{x_i} = \boldsymbol{a} + \left(\boldsymbol{i} - \frac{\mathbf{1}}{\mathbf{2}}\right)\Delta\boldsymbol{x}$ (Note: the midpoint of the interval is the average of the endpoints of the interval. For the interval [0, 1], the midpoint would be 0.5.)

Example: Let $f(x) = x^2$ on [0, 1]. Find the area below the curve using the midpoint approximation with 4 rectangles.

Dividing the area into 4 rectangles gives us the following subintervals: $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]$ where a = 0 and b = 1. $\Delta x = \frac{1-0}{4} = \frac{1}{4}$ The midpoints of each of these subintervals are found using $\boldsymbol{x_i} = \boldsymbol{a} + \left(\boldsymbol{i} - \frac{\mathbf{1}}{\mathbf{2}}\right)\Delta\boldsymbol{x}$. Since **a = 0** on this problem we will only need to use $\left(\boldsymbol{i} - \frac{\mathbf{1}}{\mathbf{2}}\right)\Delta\boldsymbol{x}$.

$$x_1 = \left(1 - \frac{1}{2}\right)\frac{1}{4} = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$$
$$x_2 = \left(2 - \frac{1}{2}\right)\frac{1}{4} = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}$$
$$x_3 = \left(3 - \frac{1}{2}\right)\frac{1}{4} = \frac{5}{2}\left(\frac{1}{4}\right) = \frac{5}{8}$$
$$x_4 = \left(4 - \frac{1}{2}\right)\frac{1}{4} = \frac{7}{2}\left(\frac{1}{4}\right) = \frac{7}{8}$$

$$A = f\left(\frac{1}{8}\right)\frac{1}{4} + f\left(\frac{3}{8}\right)\frac{1}{4} + f\left(\frac{5}{8}\right)\frac{1}{4} + f\left(\frac{7}{8}\right)\frac{1}{4}$$
$$A = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$$
$$A = \frac{1}{4}\left[\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64}\right] = \frac{1}{4}\left[\frac{84}{64}\right] = \frac{84}{256} = \mathbf{.328125}$$



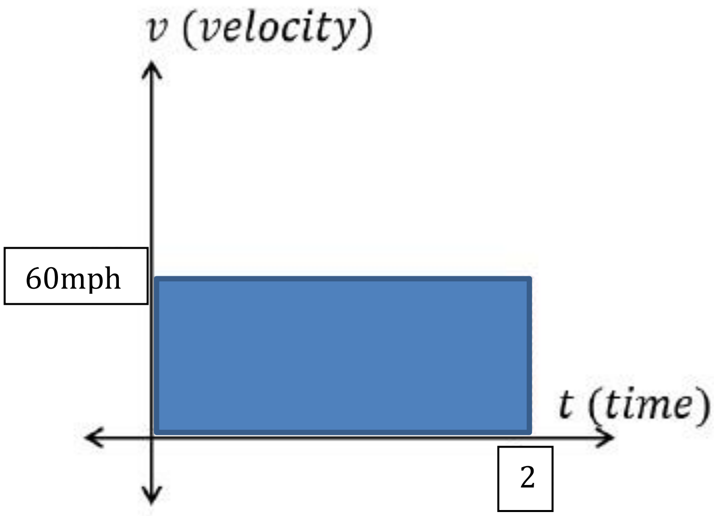
The Distance Problem

Say there is an object moving along a line with a known position function. We learned in previous chapters that the slope of a line tangent to the graph of the position function at a certain time give the velocity function of a moving object.

If there is a car traveling at a constant velocity of 60 MPH along a straight highway over a two-hour period, then the displacement of the car between **t = 0** and **t = 2** is found by:

Displacement (distance) = rate · time
= 60 MPH · 2 hr
= 120 miles

Notice that this product is the area of the rectangle formed by the **velocity curve** and the **t – axis** between **t = 0** and **t = 2**.



This is great but we know that objects do not have a constant velocity. Their velocities change over time. One thing we can do is divide the time interval into many subintervals and approximate the velocity on each subinterval with a constant velocity. Then the displacement on each interval is calculated and added. This is only an approximation, just like the area approximations we did previously. However, the approximations improve as the number of subintervals increase.

Example: Suppose the velocity in meters per second (m/s) of an object moving along a line is given by the formula $\mathbf{v = t^2}$, where $\mathbf{0 \leq t \leq 8}$. Approximate the displacement of the object by dividing the into interval **[0, 8]** into 4 subintervals. On each subinterval, approximate the velocity with a constant equal to the value of **v** evaluated at the **midpoint** of the subinterval.

Using the idea of rectangles, we have $\Delta t = \frac{8-0}{4} = 2$

$t_i = a + \left(i - \frac{1}{2}\right)\Delta t$, where a = 0

$t_1 = 0 + \left(1 - \frac{1}{2}\right)2 = 1$

$t_2 = 0 + \left(2 - \frac{1}{2}\right)2 = 3$

$t_3 = 0 + \left(3 - \frac{1}{2}\right)2 = 5$

$t_4 = 0 + \left(4 - \frac{1}{2}\right)2 = 7$

Therefore, the displacement $= f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + f(t_4)\Delta t$
$= 1^2(2) + 3^2(2) + 5^2(2) + 7^2(2)$
$= 2 + 18 + 50 + 98$
= **168** meters