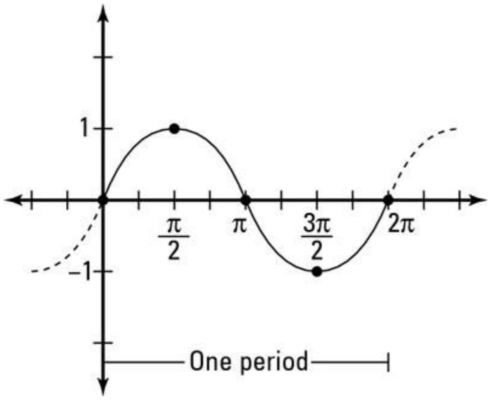
4.1 Maximum and Minimum Values

Consider the function ***f(x) = sin(x)*** for ***x*** in $[0, 2\pi]$. The plot of ***f(x) = sin(x)*** is below.

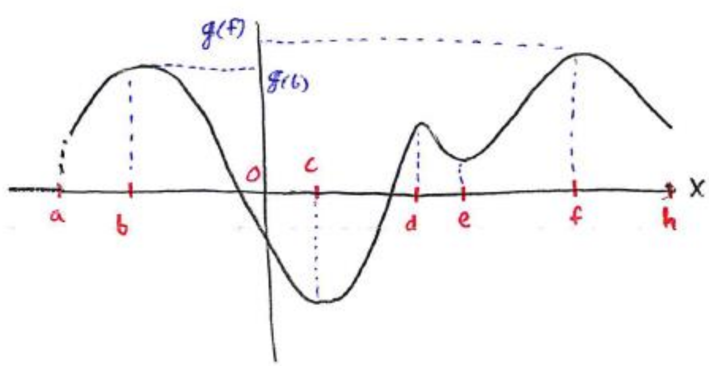


We can see that the hightest point on the graph of the function is the point $\left(\frac{\pi}{2}, 1\right)$ and the lowest value is $\left(\frac{3\pi}{2}, -1\right)$. We will say that the absolute maximum occurs at $f\left(\frac{\pi}{2}\right) = 1$ and the absolute minimum of the function occurs at $f\left(\frac{3\pi}{2}\right) = -1$. **Note:** The maximum and minimum values are the **y** coordinates.

Definition: Let ***c*** be a number in the domain ***D*** of a function ***f***. Then ***f(c)*** is the

- ***Absolute maximum*** value of ***f*** in ***D*** if ***f(c) ≥ f(x)*** for all ***x*** in ***D.***
- ***Absolute minimum*** value of ***f*** in ***D*** if ***f(c) ≤ f(x)*** for all ***x*** in ***D.***

An absolute maximum or minimum is sometimes called a ***global*** maximum or minimum. Notice that the absolute maximum and absolute minimum considers that entire domain of the function. But now what if we consider a function that has many maximum and minimum values in its domain. For example consider the following graph of function ***g***.



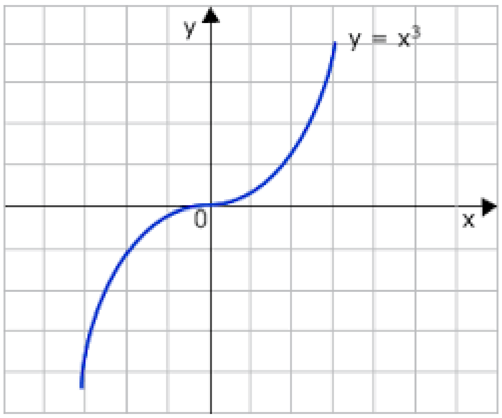
Function ***g*** has an absolute max at ***x = f*** and an absolute min at ***x = c.*** If we only consider **x-values** near ***b*** (that is we restrict the domain to be the interval (a, c)), then ***g(b)*** is the largest value in the restricted domain and we call this a ***local minimum*** value of ***g.*** Similarly, ***g(e)*** would be called a ***local minimum*** value of ***g*** because ***g(e) ≤ g(x)*** for all ***x*** near ***e*** in the restricted domain (d, f).

Definition: The number ***f(c)*** is a

- ***Local maximum*** value of ***f*** if **f(c) ≥ f(x)** when ***x*** is near ***c.***
- ***Local minimum*** value of ***f*** if **f(c) ≤ f(x)** when ***x*** is near ***c.***

A **"local"** max or min is sometimes referred to as a **"relative"** max or min.
Note: Near ***c*** means that it is true on some open interval containing ***c***. Notice that ***g(f)*** is both an absolute maximum and local maximum, whereas ***g(d)*** is only a local maximum.

Example: Consider the function $f(x) = x^3$. Find the absolute/local maximum and minimum.



This function has no maximum or minimum values. There are not absolute or local maximum or minimum values.

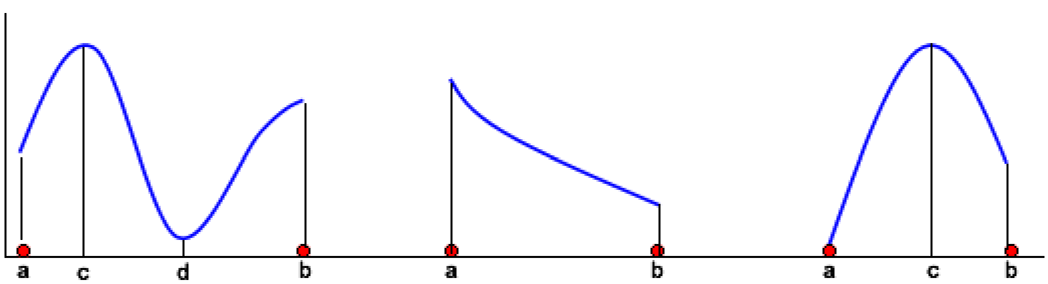
Please study example 4 on page 277.

The Extreme Value Theorem: If ***f*** is continuous on a closed interval **[a, b]**, then ***f*** has an absolute maximum value ***f(c)*** and an absolute minimum value ***f(d)*** for numbers ***c*** and ***d*** in the interval **[a, b].**

To use the Extreme Value Theorem (EVT) we need the following conditions to be true:

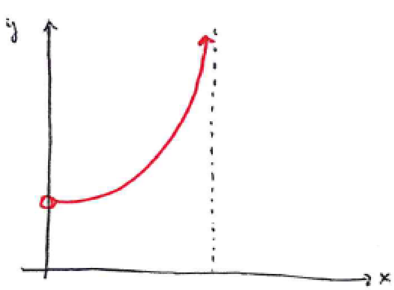
1. The function is continuous.
2. The function is defined on a closed interval.

Consider the following graphs: the domain of ***f(x)*** is [a, b].



Max at ***c***, min at ***d*** | Max at ***a***, min at ***b*** | Min at ***a***, max at ***c***.

We would not be able to use the Extreme Value Theorem in the following graph:



This function is **not** defined on a closed interval therefore the EVT does not apply.

The Extreme Value Theorem is an "existence" theorem. It tells us ***when*** there is a maximum or minimum but it does not tell us how to ***find*** the maximum or minimum, but we can find them by looking for local extreme values on the graph of a function.

If you analyze the past graphs that we have made, it appears that at the maximum and minimum points the tangent lines are horizontal and therefore have a slope of zero (0). This means that wherever the maximums or minimums occur, say at ***x = a*** and ***x = b***, then ***f '(a) = 0*** and ***f '(b) = 0.***

Fermat's Theorem: If ***f*** has a local maximum or minimum at ***c***, and if ***f '(c)*** exists, then ***f '(c) = 0.***

We must be careful when using Fermat's Theorem. Just because ***f '(a) = 0***, that does not guarantee that there is a max or min at ***a***. Fermat's Theorem suggests that we look for extreme values of ***f*** at the numbers, ***c***, where ***f '(c) = 0*** or where ***f '(c)*** does not exist. These numbers are called ***critical numbers.***

Definition: A **critical number** of a function ***f*** is a number ***c*** in the domain of ***f*** such that either ***f '(c) = 0*** or ***f '(c)*** does not exist. A function can have more than one critical number.

Example: Find the critical number(s) of the function. $\boldsymbol{f(x)} = \frac{\boldsymbol{x-1}}{\boldsymbol{x^2+4}}$

Find $f'(x)$ by using the Quotient Rule, then set $f'(x) = 0$ and solve for ***x***, the critical numbers.

$$f'(x) = \frac{(\boldsymbol{x^2+4})(\boldsymbol{1})-(\boldsymbol{x-1})(\boldsymbol{2x})}{(\boldsymbol{x^2+4})^2} = \frac{x^2+4-2x^2+2x}{(\boldsymbol{x^2+4})^2} = 0 \;\Rightarrow\; x^2+4-2x^2+2x = 0 \;\Rightarrow\; -x^2+2x+4 = 0$$

Use the quadratic formula to solve: $x = \frac{-2\pm\sqrt{4-4(-1)(4)}}{-2} = \frac{-2\pm\sqrt{20}}{-2} = \frac{-2\pm2\sqrt{5}}{-2} = \boldsymbol{1\pm\sqrt{5}}$ Since this function is defined for everywhere $(-\infty,\infty)$ then both solutions for ***x*** are critical numbers. If you look at this function on your graphing calculator (viewing window: x[-10, 10] x-scale: 1, y[-1, 1] y-scale: 1) you should see that when $x = 1+\sqrt{5}$ there is a maximum and when $x = 1-\sqrt{5}$ there is a minimum.

We can also rephrase Fermat's Theorem to be:

If ***f*** has a local maximum or minimum at ***c***, then ***c*** is a critical number of ***f.***

The **Closed Interval Method [a, b]:** To find the absolute maximum or minimum values of a continuous function ***f*** on a closed interval [a, b]:

1. Find the values of ***f*** at the critical number of ***f*** in the open interval (a, b).
2. Find the values of ***f*** at the endpoints of the closed interval [a, b].
3. The largest values from steps 1 and 2 are the absolute values; the smallest of these values are the absolute minimum values.

Example: Find the absolute minimum and/or maximum values of ***f*** on the given interval. $f(x) = x - \sqrt[3]{x}$ on the interval [-1, 4]

1. Find the critical numbers of the function. $f'(x) = 1 - \frac{1}{3}(x)^{-\frac{2}{3}} \;\Rightarrow\; 1 - \frac{1}{3\sqrt[3]{x^2}} = 0 \quad 1 = \frac{1}{3\sqrt[3]{x^2}}$

$3\sqrt[3]{x^2} = 1 \quad \Rightarrow \quad \sqrt[3]{x^2} = \frac{1}{3} \quad \Rightarrow \quad x^{\frac{2}{3}} = \frac{1}{3} \Rightarrow x = \left(\frac{1}{3}\right)^{\frac{3}{2}} \quad \Rightarrow \quad \boldsymbol{x} = \frac{\sqrt{3}}{9}$ **critical number**

2. Find the values of the function at the critical numbers and at the endpoints of the interval.

$f\left(\frac{\sqrt{3}}{9}\right) = \frac{\sqrt{3}}{9} - \sqrt[3]{\frac{\sqrt{3}}{9}} \quad \cdots = -\frac{2\sqrt{3}}{9} \qquad f(-1) = -1 - \sqrt[3]{-1} \cdots = 0 \quad f(4) = 4 - \sqrt[3]{4}$

$\approx -.3849001795 \qquad\qquad \approx 2.412598948$

The absolute minimum occurs at $\boldsymbol{x} = \frac{\sqrt{3}}{9}$ and the absolute minimum **value** is ≈ -0.385. The absolute maximum occurs at $\boldsymbol{x} = \boldsymbol{4}$ and the absolute maximum **value** is ≈ 2.413.